REVISED EDITION

College Algebra

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HARPER & BROTHERS PUBLISHERS

COLLEGE ALGEBRA

FIRST EDITION

B—P

CONTENTS

EDITOR'S PREFACE TO THE FIRST EDITION

[As a tribute to the distinguished editor of the first edition of the books of this series, this preface is retained in this revised edition exactly in its original form, though the page references and other minor parts are not directly applicable to the present book. It is a great loss to the teaching of mathematics that his services are no longer available.]

In some respects the making of a good college algebra is more difficult than that of almost any other college text in mathematics. The topics are largely disconnected and vary greatly in difficulty. The logical basis of algebraic reasoning is so apparently different from that which the student has met in geometry and trigonometry that the author may well hesitate as to the extent to which he will stress this phase of presentation. I believe this text has met these difficulties in unusually effective ways. For instance, the topics forming the usual college course are found in a continuous presentation beginning with the discussion of the quadratic function (page 81) and ending with the chapter on complex numbers (page 180). This compact body of material, in sufficiently logical form, follows eighty pages of review of elementary topics to be used as needed by the weaker students and those of inadequate preparation, and is followed by about fifty pages of various topics discussed in more rigorous logical form appropriate for use by the brighter students who are able to stand the greater mental stimulus. This latter provision for the abler students, which is now recognized as highly important, has been worked out in this text with no little skill and should be of great assistance to the up-to-date teacher.

Two other matters in the text are worthy of note. One is the cumulative reviews to which the author has given much careful attention and which are of prime importance as set forth by him in his preface. The other is the extensive and comprehensive historical sketch which should add greatly to the usefulness of the text both in the class room and in the freshman mathematical club—assuming, as always, that such a club should exist and function effectively in every well-organized collegiate institution.

The University of Chicago.
September, 1928

H. E. SLAUGHT

AUTHOR'S PREFACE

While this book appears as a Revised Edition, it is really a new book. Only a few pages remain as in the first edition and by far the greater part has been completely rewritten.

Parts of the old book were written while I was instructor at the Massachusetts Institute of Technology and in Columbia University. Since then the American undergraduate scene has undergone great changes—changes which at the time that the first edition was finally completed were well under way, but which at that time were not fully recognized by myself or by the editor of this series. The purpose of the present edition is, naturally, to meet conditions as they now exist, and to provide for the varied demands that our present freshman classes make upon the college and the university.

Uneven preparation of entering students. Among those who now enter mathematics classes in the college, a large number have had only one year of secondary school algebra and one year in plane geometry, while a smaller number have had a year and a half of algebra, and a still smaller number have had four years of mathematics, two of which were devoted to algebra. These students are often, perhaps usually, taught in different classes and on different levels; but it is by no means rare to see pupils of such mixed attainments brought together in one group to study a not too sharply defined subject called college algebra. It is obvious that if the same text is to meet the needs of such varied classes, its more elementary parts must be expanded very considerably beyond the traditional limits. This will be found to be conspicuously true of the present book. The purpose has been to adapt the first part of the book to the needs of students who have had only one year of secondary school algebra, while at the same time shading the work into a level requiring somewhat greater maturity than in a second course in ordinary high school algebra. While this is true of the first part of this book, it is believed that the more advanced parts are fully adequate for the needs of the most advanced classes that normally study a first course in college algebra.

Increased emphasis on "investments" and on probability. The rapidly increasing number of students in our classes dealing with the Mathematical Theory of Investments seems to make it advisable to include a more detailed treatment of accumulation and discount at compound interest, and of the values of annuities at the beginning and the end of the term of an annuity. The purpose is not to replace a course on investments, but to use the geometric series in obtaining the formulas $S = \dfrac{(1+i)^n - 1}{i}$ and $A = \dfrac{1-(1+i)^{-n}}{i}$, which represent the main mathematical elements in the theory of investments. The theory of probability is developed somewhat more fully than is customary and some applications are found in elementary problems on insurance, data from the American Experience Table of Mortality being used.

Work for the abler students. One of the most pressing of the special needs that I have come to feel is an arrangement making it easy and convenient to give the abler students the kind of work they should have. Provision is made here for such work by making the later parts of the various chapters somewhat more difficult, while at the same time giving an abundance of exercises and problems for the median students without including these more exacting parts. Again, such chapters as those dealing with the solution of the cubic and the biquadratic, with partial fractions, and some of the theorems on the theory of equations that are not needed in using Horner's Method, may be omitted in the regular course but may be used as a reservoir of work for the abler students.

The cumulative reviews. Much has been said and written in deprecation of "mere memory work." But what is deprecated is not the remembering, but memorizing without understanding. Obviously remembering is not only desirable but necessary. It is precisely in this field of memory that the investigators in psychology during the last sixty years have obtained some of their most significant and best-substantiated results. Vivid initial impressions, and understanding of the meaning of that which is learned, shorten the time of learning and increase retention; this is a matter of common sense (and is confirmed by laboratory experiments), and account of it is no doubt taken very generally in teaching.

But there is in this connection another fact which, I believe, is not taken into account very generally; certainly our texts make no provision for taking account of it. We are always forgetting, and to secure relatively permanent retention a series of relearnings is necessary. This has been affirmed by the results of experiments without number. We study a topic and make it fairly clear to ourselves; but in a crowded life it is laid aside and in a few days is forgotten, so that without prompting we cannot recall any of it. Restudy, however, is very much easier than the first learning. Each successive restudy becomes easier than the preceding, and after a series of such restudies retention becomes relatively permanent. No fact in connection with learning is more important than this, and no fact has been neglected more consistently. The teacher who has relearned his subject so many times finds it hard to understand how naturally and completely the student forgets. The usual procedure is a review (a single relearning) just before the final examination, and after that there is undisturbed oblivion.

The Cumulative Reviews in Chapter 26 of this book make provision for systematic relearnings. There are forty-six groups of exercises, six in each group, and the plan is to assign these groups in order, one each day. The subject matter of the text has been analyzed and the exercises in the Cumulative Reviews are so spaced that each topic recurs approximately with the proper spacing and frequency to conform to the "curve of forgetting and relearning."

In a strong class in which the more elementary parts of algebra are reviewed quite sketchily, these cumulative reviews may be assigned from the beginning. They will serve to recall those simple parts of algebra that are in constant use but about which time has begun to cast its ever thickening mist.

On the whole these review exercises are quite simple, while at the same time they contain some elements of novelty so that they will provoke *thoughtful* relearning.

However, the main body of this text is exactly as it would be if these reviews were not in the book. In my own practice the study of the course proceeds as if they were not in existence and they are assigned each day as a sort of a side issue; but they do make for a type of retention that without them is simply impossible. For one thing they make a review of the usual type unnecessary. Instead, a final review may become a consideration of relations between parts now well known and of the general bearing and significance of the whole subject. It is my belief that the systematic use of such reviews is the single greatest general improvement that is immediately and easily possible in our teaching, and that the neglect of them is one of the two greatest pedagogical sins we have committed. The other is the neglect of the gifted student.

The historical sketch. A fairly extensive historical sketch is printed at the end of the book. Rarely does a student go to the treatises for such material, and even when he does he finds it no small task to pick out the account of his particular subject. If the story of a subject such as algebra is rightly read, it is an interesting and illuminating chapter in the history of culture; it illustrates the fairly universal difficulty that man has had in making new discoveries and the frequent slowness with which discoveries are recognized after they are once made. The plan of inserting "historical notes" in the text has often been used, but such notes cannot tell a connected story, and seem to me decidedly inferior in value when compared with a single unified historical account.

Abundance of exercises and problems. It is becoming increasingly clear that the teachers in our schools and colleges feel a distinct need of an abundance of exercises and problems so that the student may have the opportunity of learning by actually using the theoretical part that contains the real substance of his work. But this "substance" may well remain inert and meaningless unless it is given life through use. For this reason the present text will be found to contain an exceptionally rich body of exercises and application in verbally stated problems.

The revised edition owes much to teachers who have been good enough to criticize the older book. I am especially indebted to those who have confirmed my own judgment that the book was, in part, too sketchy. This fault has, I believe, been fully removed in this rewritten edition.

N. J. Lennes

COLLEGE ALGEBRA

CHAPTER 1:

THE REAL NUMBER SYSTEM OF ALGEBRA

Our ordinary numbers and numerals are used in algebra with exactly the same meaning that they have in arithmetic. The purpose of this chapter is to describe these a little more fully, and then to introduce an important extension of the numbers of arithmetic, which is needed at the very beginning of algebra. (See page 9, §10)

1. *Positive integers.*—The words "one," "two," "three," . . . that we say when counting are "number words," or numbers, and the symbols 1, 2, 3, . . . that represent these words are our ordinary whole numbers, or positive integers. The symbol 0, zero, is often included in the system of positive integers. The positive integers are also called the natural numbers.

It has been argued that zero represents the absence of number and hence cannot be regarded as a number. However, no contradiction results from regarding it as a part of the group of things we call numbers, as is now the general usage.

2. *Cardinal and ordinal numbers.*—If a positive integer is used to represent the number of objects in a group, such as the number of fingers on a hand, then it is called a cardinal number. But a positive integer may also be used to indicate the position of an object in an ordered sequence as the "first," the "second," the "third," and so on. When an integer is used for this purpose, it is called an ordinal number.

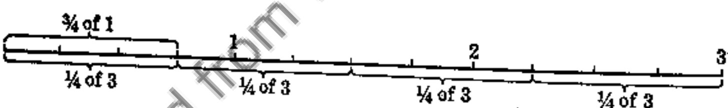
3. *Inequalities of integers.*—If a and b represent positive integers and if a precedes b in the sequence of natural numbers, then a is less than b, and this is indicated by $a<b$, read "a is less than b." In this case b is greater than a, and this is indicated by $b>a$. Both of these are of course indicated by the one symbol $a<b$, or by the one symbol $b>a$.

4. ***Positive fractions: the first extension of the number system.***—Historically, man's first acquaintance with numbers was naturally with ordinary integers, as is also that of every child. Beginning with these "natural" numbers, the number system has undergone a series of extensions, the first and simplest of which was the development of ordinary positive fractions.[1] As counting led to the "natural" numbers, so measuring led to the use of parts of a unit, that is, to fractions. The idea of a fraction as an indicated division came much later, but the great convenience of this idea in working with mathematics has led to its general use. We shall therefore adopt the following definition.

An indicated quotient of two integers when written in the form m/n *is called a rational fraction.*

With this definition of a fraction and also the definition of division, n/m is a number such that $m \times n/m = n$. Thus $4 \times 3/4 = 3$. This means at once that 3/4 is one of the 4 equal parts of 3. This is verified directly by using a figure as is shown below.

By looking at this figure it is seen that the fraction ¾ may be regarded as 3 divided by 4, that is, as ¼ of 3; or it may be regarded as 3 of the 4 equal parts of a unit.



In general, m/n (n not equal to 0) may be regarded as m of the n equal parts of unity or as the quotient obtained by dividing m into n equal parts. These are the ordinary fractions of arithmetic. In the fraction m/n, m is the numerator of the fraction and n is its denominator. The numbers m and n taken together are the terms of the fraction. The fraction m/n is also called the ratio of m over n.

If in m/n, m is less than n, then m/n is a proper fraction; otherwise it is an improper fraction. By dividing, an improper fraction may be reduced to the sum of an integer and a proper fraction, or possibly to an integer.

[1] For the purpose of this chapter we shall assume informally the simple laws of arithmetic on positive integers and fractions, and also the idea of "less than" and "greater than" when applied to these numbers.

The numerator and denominator of a fraction are respectively the dividend and divisor of an indicated division. For the case of an improper fraction we then have:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

This indicates that the denominator is contained a whole number of times (the quotient) in the numerator, with a remainder (possibly zero) that is still to be divided.

The integers and fractions of arithmetic taken together form the rational numbers of arithmetic. The rational number system has the interesting property that in it there is no number next to another number. Thus there is no fraction that is next to the number 1. For instance, 87/88 is less than 1 but it is not "next" to 1 since 97/98 is also less than 1 but greater than 87/88. In general, if m/n is less than 1, then it is easy to verify that $(m + 1)/(n + 1)$ is between m/n and 1. In the same manner it is easily seen that if a and b are any two rational numbers, then $(a + b)/2$ is a rational number between a and b.

EXERCISES

1. Make figures to show two different meanings of each of the fractions 2/3, 3/5, 5/6.

2. If a and b are rational numbers, that is, if $a = m_1/n_1$ and $b = m_2/n_2$, show that $(a + b)/2$ is a rational number.

3. If $(0 + 1)/2 = q_1$, $(q_1 + 1)/2 = q_2$, $(q_2 + 1)/2 = q_3$, . . . , what are the values of $q_1, q_2, q_3, \ldots$? As you go on in this sequence how are these quotients related to the number 1?

4. If $(2 + 3)/2 = q_1$, $(2 + q_1)/2 = q_2$, $(2 + q_2)/2 = q_3$, . . . , find four successive values of $q_1, q_2, q_3, \ldots$ How are these quotients related to the number 2?

5. If $a<b$, and if $(a + b)/2 = q_1$, $(q_1 + b)/2 = q_2$, $(q_2 + b)/2 = q_3$, . . . , describe the sequence of numbers $q_1, q_2, q_3, \ldots$ For any one of these numbers is it greater or less than the one immediately preceding it? the one immediately following it? How are all these numbers related to a and b?

6. What light do examples 2–4 throw on the statement that in the system of rational numbers there is no number "next" to the number 1 or the number 2?

5. *The decimal number system.*—For practical purposes it would be impossible to use independent number-words even for all the whole numbers that we need to use; one thousand of them would be very difficult to learn and remember as standing in a definite order. For this reason a system something like our decimal numbers is a necessity. Using the decimal system we count to ten, and then start over again, using ten and one, ten and two, and so on up to ten and ten, which we call twenty; then twenty and one, twenty and two, up to twenty and ten, which we call thirty; and so on. The fact that we use "ten and one," "ten and two," etc., is partly concealed by the use of the special words "eleven," "twelve," and so on up to "nineteen." From twenty on, the words used show clearly what is meant, though the word "and" is omitted, as in "twenty-one" for "twenty and one."

Ten tens is called one hundred, ten hundreds is called one thousand, one thousand thousands is called a million, one thousand millions is called a billion. Hence we see that twenty-five independent number words (including "zero," and "eleven," "twelve," up to "nineteen," but not the obvious contractions twenty, thirty, up to ninety) are sufficient for the naming of all whole numbers that come into practical use.

For reasons that are apparent the number ten is called the base of the decimal number system. The reason that ten, and not some other number, came to be used as the base is supposed to be that primitive man counted on his fingers. Counting on the fingers of both hands gives ten. Then starting over again, we get ten and ten, or twenty, and so on. It has even been surmised that two men (or possibly even more) did the counting. One man counted to ten on his fingers and then the other man held up one finger to indicate one ten. The first man now started over again and when he reached ten the second time, the other man held up two fingers, and so on.

It has been proposed to change our number system to some other base than ten, to twelve for instance, and to invent for that purpose new single symbols for 10 and 11. This would have the advantage that 12 is divisible by 2, 3, 4, and 6, while 10 is divisible by only 2 and 5. However, we have become accustomed to the base 10, and there is an immense printed literature and body of records of which our present system is a definite and important part. This would make such a change extremely difficult and expensive, and we are likely to go on indefinitely with our present system. In mathematical discussion, 2 is sometimes used as a base.

6. *The Arabic system of numerals.*—The system of numerals that we use, called the Arabic system, has two characteristics that make it superior to any other system that has come into general use.

1. One of these is the principle of place value, whereby the value of a numeral depends upon the place in which it stands, as in the following.

444 = 4 hundreds + 4 tens + 4 ones
1234 = 1 thousand + 2 hundreds + 3 tens + 4 ones

In the number 1234, 4 is said to be in ones' place, 3 in tens' place, 2 in hundreds' place, and 1 in thousands' place.

2. The second characteristic is that when there is no such numeral (1, 2, 3, 4, 5, 6, 7, 8, 9) to go into a place, then this place is filled with a zero (0), as in the following.

4070 = 4 thousands + no hundreds + 7 tens + no ones

The symbol 0 is now a regular part of our system of numerals (see page 1, §1) and is usually said to belong to the system of integers. That is, the integers of arithmetic are

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, . . .

In this way every integral number is written by using the ten numerals from 0 to 9 inclusive. These numerals are also called digits.

TOPICS FOR DISCUSSION

1. How did measuring lead to the invention of fractions? Could ordinary counting lead to this invention?

2. Did the Romans use the principle of place value in their system of numerals? Would the number zero be likely to be invented by people who did not use the principle of place value?

7. *Decimal fractions.*—The extension of the Arabic system of numerals to include decimal fractions is obvious when once discovered. Thus .444 = 4 tenths + 4 hundredths + 4 thousandths. In any number expressed in the Arabic system of numerals we have the following general property.

Any digit in a decimal number has ten times the value of the same digit in the next place to the right.

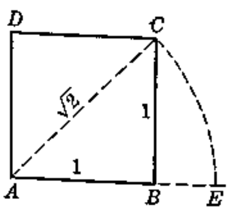
We must distinguish clearly between the decimal system of numbers and the Arabic system of numerals. The former was in use thousands of years before the invention of the latter. It was used, for instance, by the Babylonians, the Egyptians, the Greeks, and the Romans, none of whom, with the exception of the Babylonians, had even an inkling of the Arabic system of numerals.

EXERCISES

1. Does the principle of place value apply to such a number as 8000? Does it apply to the number 30.04006?

2. Write the numbers 317 and 5286 in Roman numerals and then try to add them or find their product. This will help to show the importance of the Arabic system of numerals.

8. *Irrational numbers; the second extension of the number system.*—It is easily shown that the length of a diagonal of a square whose sides are unity cannot be represented exactly by a rational number. However if a square is constructed as in the figure and the arc CE having A as its center is struck, then the length of AE thus laid off is the side of a square whose area is 2. Hence AE is the square root of 2, while the length is not a rational number. The square root of a number a is indicated by $\sqrt{a}$.



Numbers such as the square root of 2 are called irrational numbers. Irrational numbers are also called incommensurable numbers.

If the indicated root of an integral number is not an integer, then it is an irrational number. Thus $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[3]{4}$ are irrational numbers, since it is easy to prove that such numbers cannot be rational fractions. There is, for instance, no rational fraction m/n

such that $m^2/n^2 = 2$, or 3; nor is there a fraction such that $m^3/n^3 = 4$. Besides such indicated roots there is a vast number of other irrational numbers. Thus the number π (3.141592 . . .), the ratio between the circumference and diameter of a circle, which is encountered in plane geometry, is an example of an irrational number that is not an indicated root of a rational number.

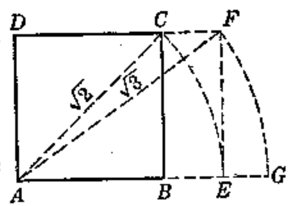
In practice irrational numbers are approximated in decimals to any required degree of closeness. The square root of 2 is approximately 1.4, or 1.41, or 1.414, or 1.4142, or 1.41421, . . . , the number of decimals used depending upon the nature of the problem with which we are working. The examples below will show, for instance, that the square root of 2 lies between 1.41421 and 1.41422. Since every measurement is only an approximation, every number thus resulting can be expressed entirely adequately by an ordinary decimal. That is, irrational numbers find no direct practical application. However, they make for such great simplicity and smoothness in mathematical theory that they are highly important.

EXERCISES

1. Square (multiply by itself) each of the numbers 1.4, 1.41, 1.414, 1.4142, 1.41421. By how much does each of these squares differ from 2?

2. Increase by 1 the last digit of each of the numbers in example 1, obtaining 1.5, 1.42, ... Square these numbers. By how much does each of these squares differ from 2?

3. Show that in the figure at the right the length of AC (or AE) is $\sqrt{2}$, that the length of AF (or AG) is $\sqrt{3}$. Compare the figure on page 6, §8.



4. Construct in sequence $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$. Which numbers in this sequence are rational numbers?

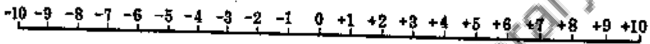
5. If a diagonal of a rectangle is $\sqrt{5}$ and one of its sides is unity, what is the area of the rectangle?

6. In the system of integers is there a largest number? a smallest number? Is there an integer between any given pair of integers? In the system of ordinal numbers is there a first? a last?

7. In the system of positive numbers (including the irrationals) is there a smallest number? a largest? Does your answer depend upon whether you regard zero as a positive number? Is there a number that is "next" to any given number?

9. ***Signed numbers: the third extension of the number system.***—In many applications of numbers to practical problems it is convenient, and in some cases necessary, to start at some point and measure from it in opposite directions. We usually label this starting point zero and then use numbers with a + sign on one side and numbers with a − sign on the other. Thus we have temperatures above zero, as +65°, and temperatures below zero, as −15°.

Such a method is absolutely necessary when we attempt to lay off a "scale" on an endless straight line. We cannot start numbering the division points at one end of the line since it has no end.



This figure shows the scale of signed numbers. The point marked 0 on this scale is often called the origin.

We start at any point on the line, labeling it 0, and lay off equal units in both directions, labeling the end points +1, +2, +3, . . . to the right and −1, −2, −3, . . . to the left. The numbers with the + sign are called positive numbers and those with the − sign are called negative numbers. The + sign is often omitted before a positive number. Thus 3 is "positive 3" or simply 3, while −3 is "negative 3."

Fractions may be developed for negative values in the same manner that positive fractions are developed. Measuring a distance in the negative direction we may obtain any fractional length, which will then be negative. This method can be extended to the irrational numbers. If we have a line segment whose length is $\sqrt{2}$, we may then lay off this segment in the negative direction starting at the origin. The left end point of this segment represents the irrational number $-\sqrt{2}$. Clearly, any negative irrational number, such as $-\pi$, can be given definite meaning in this way. This completes the real number system of algebra. (See page 9, §10.)

TOPICS FOR DISCUSSION

Without attempting any proofs answer these questions: May the complete real number system of algebra be represented on a "scale" of signed numbers? If a and b are any two real numbers, must one and only one of statements $a = b$, $a > b$, $a < b$ be true?

10. The real number system of algebra.—The positive and negative numbers described above, when taken together, form the real number system of algebra. This number system is also called a signed or directed system. The positive numbers may be regarded as the ordinary numbers of arithmetic. When the sign of a number is disregarded, its value is called the absolute or numerical value of the number. Thus 3 is the absolute value of both -3 and $+3$. When a letter, such as a, is used to represent a number, then $|a|$ indicates its absolute value. Thus $|-3| = |+3| = 3$.

Starting with ordinary integers, three extensions, or generalizations, of the number system have now been made: *1.* The ordinary fractions, 2. irrational numbers, 3. negative numbers. The first of these extensions is always made in ordinary arithmetic. The second extension is also made, implicitly, in arithmetic when working with square roots of numbers. The third extension is entirely peculiar to algebra.

To complete the number system of algebra one more extension will be made in Chapter 10. As we proceed we shall see that on the one hand the theoretical requirements of algebra (in part also of arithmetic) demand these extensions and that on the other hand they are needed in the practical applications of mathematics.

EXERCISES

1. What kind of number system is needed if the numbers are used to label the points on a complete straight line? Could the origin be put at one end of this line? Are fractions necessary for this purpose? Are irrational numbers necessary? (See page 6, §8.)

2. If numbers are to be applied to time, past, present, and future, what kind of number system must be used?

3. Name a number of situations in which the signed number system is necessary or convenient.

4. If the number -3 represents a point on the scale of signed numbers, exactly what is represented by $|-3|$? What does the sign $-$ in -3 represent? In this case does the $-$ sign indicate subtraction? If $-7°$ represents a temperature reading, what does $7°$ represent? What does the $-$ sign represent? In this case does the $-$ sign indicate subtraction? Is a temperature below zero to be subtracted any more than a temperature above zero?

CHAPTER 2:

LAWS OF THE FUNDAMENTAL OPERATIONS

The operations of addition, subtraction, multiplication, and division are called the fundamental operations of arithmetic and algebra. When dealing with these from a little more advanced point of view than that adopted in the beginnings of these subjects, it is customary *not* to define addition and multiplication, but to adopt certain axioms stating the laws according to which they are performed. In this book we shall depart from this usage to the extent of pointing out in some detail the nature of the fundamental operations, and showing a basis for the axioms that will finally be adopted.

11. Addition.—Suppose we have two groups of objects, the first containing a objects and the second b objects. To find the total number of objects in the two groups we may count them. If we know there are a objects in the first group, we may start counting the objects in the second group, saying $a + 1$, $a + 2$, $a + 3$, and so on, until the second group has been exhausted. The last number in this counting will be the sum of a and b, which we indicate by $a + b$. The numbers a and b that are added are called addends. A fundamental property of addition is that two numbers have one and only one sum. We speak of this property as the uniqueness of addition.[1] The process of finding the sum is called addition.

In practice, addition is performed without counting. We first learn the sums of small numbers, such as $7 + 8 = 15$, and then organize the usual processes of arithmetic for finding the sums of larger numbers or of several numbers. Addition of fractions, of signed numbers, and of irrationals is then built up step by step. This can be done by a process very similar to that used for integers. In m/n, the fraction $1/n$ may be regarded as a unit, when the number m of these units may be counted exactly as in the case of integers.

[1] At this point it is intended to assert only that if a and b are added in the order a b, the sum will always be the same. Other statements about addition are given in the sections that follow.

12. The commutative law of addition.—In adding 6 and 4 we may start with 6 and add 4 to it (indicated by $6 + 4$), or we may start with 4 and add 6 to it (indicated by $4 + 6$). The sum in each case is 10. It is easy to give a fairly clear-cut proof that counting the objects in a group will give the same number no matter in what order the objects are counted. This means that when counting the objects in two groups, containing a and b objects respectively, we may start with a and then count the objects in the second group until these are exhausted, obtaining the sum $a + b$; or we may start with b and count the objects in the first group, obtaining the sum $b + a$. Since each of these sums is the number of objects in the two groups when taken together, we see that $a + b = b + a$. This is called the commutative law of addition. It says that numbers to be added may be "commuted" or arranged in any order without changing the sum.

$$6 + 4 = 4 + 6$$
$$a + b = b + a$$

13. The associative law of addition.—In adding 6, 4, and 2, we may start with 6, then add 4 to it, and finally add 2 to this sum. This is what is indicated by the expression $6 + 4 + 2$. But we may add 4 and 2 first and then add the sum to 6. This is what is indicated by $6 + (4 + 2)$. The final sums will be the same in the two cases. Note that $6 + 4 + 2$ and $(6 + 4) + 2$ indicate exactly the same order of steps.

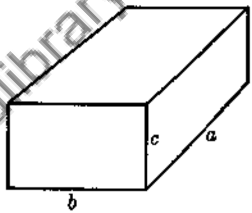
$$6 + 4 + 2 = 6 + (4 + 2)$$
$$a + b + c = a + (b + c)$$

If we have three groups of objects containing a, b, and c objects respectively, then we may find the total number of objects in these groups by finding the number $a + b$ of objects in the first two groups and then counting the objects in the third group, saying $a + b + 1$, $a + b + 2$, . . . until the objects in the third group are exhausted, thus obtaining the sum $a + b + c$. But we may find the sum $b + c$ first and then add this sum to the number in the first group, obtaining $a + (b + c)$. It follows that $a + b + c = a + (b + c)$. This is called the associative law of addition, or associative law of addends. It says that numbers to be added may be "associated," or grouped, in any manner without changing the sum.

17. *Associative law of factors.*—In finding the product $2 \times 3 \times 4$, we may multiply 3 by 2 and then multiply 4 by this product. That is, $2 \times 3 \times 4 = 6 \times 4$. But we shall obtain the same product if we first multiply 4 by 3 and then multiply this product by 2. That is, $2 \times 3 \times 4 = 2 \times 12$. Taking the factors in this way is indicated by $2(3 \times 4)$. This is a special case of the associative law of multiplication, or the associative law of factors.

$$2 \times 3 \times 4 = 2(3 \times 4)$$
$$abc = a(bc)$$

This law may be illustrated by considering a rectangular solid a units long, b units wide, and c units deep. We may consider this solid as made up of c layers, each containing ab unit cubes. Hence the number of cubes is $c(ab) = (ab)c = abc$, [$c(ab) = (ab)c$ by page 13, §16]. But we may also regard it as made up of a layers (slices) each containing bc unit cubes or $a(bc)$ cubes. Hence $abc = a(bc)$. Note that abc and $(ab)c$ indicate exactly the same steps in multiplying.



Again we may regard this solid as consisting of b layers (slices) each consisting of ac unit cubes. Hence $abc = b(ac)$.

EXERCISES

1. Construct a rectangle 6 units wide and 9 units long, and show by means of it that $6 \times 9 = 9 \times 6$.

2. A rectangular solid is 5 units wide, 12 units long, and 10 units high. In what ways can you find the number of cubic units in this solid?

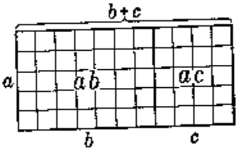
3. The dimensions of a rectangular solid are 4, $5\frac{3}{5}$, $6\frac{3}{4}$. Find the volume of this solid by multiplying the dimensions in different orders. Which order makes the computation the easiest?

18. *The distributive law of multiplication with respect to addition or subtraction.*—It is easily verified that to multiply the sum of two numbers we may multiply thus: $6(7 + 3) = 6 \times 7 + 6 \times 3 = 42 + 18 = 60$, and also: $6(7 + 3) = 6 \times 10 = 60$. Similarly, $6(7 - 3) = 6 \times 7 - 6 \times 3 = 24$. In general, $a(b + c) = ab + ac$, and $a(b - c) = ab - ac$. This is called the distributive law of multiplication.

$$6(7 + 3) = 6 \cdot 10 = 60$$
$$6(7 + 3) = 6 \cdot 7 + 6 \cdot 3 = 42 + 18 = 60$$
$$a(b + c) = ab + ac$$

$$6(7 - 3) = 6 \cdot 4 = 24$$
$$6(7 - 3) = 42 - 18 = 24$$
$$a(b - c) = ab - ac$$

This law is further illustrated by using the figure below. The number of unit squares in the whole figure is obviously $ab + ac$. But we may also regard it as made up of a strips, each $b + c$ long, and hence the number of squares is $a(b + c)$. That is, $a(b + c) = ab + ac$. In this manner $a(b - c) = ab - ac$ may be illustrated by using a figure in which the length represents $a - b$.



When we compare the commutative and associative laws for addition and multiplication, we see that they are exactly alike, with the exception that "multiplication" is used in one statement where "addition" is used in the other. However, the distributive law of multiplication with respect to addition serves to distinguish between addition and multiplication. That is, to multiply a sum we may multiply the addends separately and then collect the products. But we cannot interchange "addition" and "multiplication" in this statement. That is, to add a number to a product, we may not add the number to each factor and then collect the sum.

EXERCISES

1. Construct a figure similar to the above to show that $a(b - c) = ab - ac$.

2. Each of two rooms is 19 feet wide. One room is 23 feet long and the other is 27 feet long. Find in two ways the number of square feet of flooring required for these rooms. Which way is the easier? By what law are you permitted to choose either method?

3. From a board 16 feet long and 10 inches wide a piece $8\frac{3}{4}$ feet is cut off. How many square inches are there in the remaining piece? Find the answer in two ways. What law do you use?

4. At \$1.25 (\$5/4) per cubic yard find the cost of making an excavation 36 feet wide, 45 feet long, and 6 feet deep. Suggestion: Reduce dimensions to yards. Multiply in several different orders. Which orders avoid fractional products? Note that the commutative and associative laws of multiplication apply to any number of factors.

5. In example 2 the cost of the flooring is $31\frac{1}{2}$ cents a square foot. Find in different ways the cost of the whole flooring. Compare the trouble of computing when using the different ways.

6. Compare the commutative and associative laws of addends with these laws for factors.

19. Division.—*Dividing a number* a *by a number* b *is the process of finding a number* q (*quotient*) *such that* $bq = a$. Note that division may lead to a quotient which is a fraction, though we start with the integers a and b. In this respect it differs from the other three fundamental operations, since the sum, difference, and product of two integers is always an integer.

When mathematics is considered from a certain point of view, fractions are regarded as being "invented" in order to make possible the division of any number by any other number (except by 0).

The following definitions show how similarly subtraction and division are related respectively to addition and multiplication.

Subtraction is the process of finding one of two numbers when their sum and the other number are given.

Division is the process of finding one of two numbers when their product and the other number are given.

Subtraction and division are the inverses of addition and multiplication respectively.

20. Zero; unity.—The number zero, 0, has the property that adding it to any number a gives a as the sum. Also, subtracting 0 from any number a gives a as the remainder. That is, using 0 as an addend or subtrahend does not change the result. Subtracting any number a from itself gives 0 as the remainder.

$$a + 0 = a$$
$$a - 0 = a$$
$$a - a = 0$$

Unity has properties as related to multiplication and division that are strictly analogous to those of zero as related to addition and subtraction. That is, multiplying any number a by unity, 1, gives a as the product, and dividing a by 1 gives a as the quotient, while a divided by a gives unity as the quotient.

$$1 \cdot a = a$$
$$a/1 = a$$
$$a/a = 1$$

From the distributive law of multiplication, $a(b - c) = ab - ac$, we have $a(b - b) = ab - ab = 0$. That is, multiplying any number a by 0 ($b - b = 0$) gives zero as the product. It follows that division by 0 is either impossible or completely indeterminate. Thus, if $0/0 = q$ (q for quotient), then $0 = 0 \cdot q$. But this holds for any value of q. Hence $0/0$ is equal to any number whatever, and this symbol cannot be used.

$$0/0 = q$$
$$0 = 0 \cdot q$$

Again, if we let $a/0 = q$, a being different from zero, then $a = 0 \cdot q$ and hence $a = 0$, which is a direct contradiction. Therefore if a is different from zero, then $a/0$ has no value whatever. For these reasons division by zero is definitely excluded in both arithmetic and algebra. Many errors have crept in because of failure to observe this rule.

$$a/0 = q$$
$$a = 0 \cdot q$$
$$\therefore a = 0$$

The properties of 0 shown at the right are obvious.

If $ab = 0$
then $a = 0$
or $b = 0$
$0/a = 0 \cdot 1/a$
$= 0$

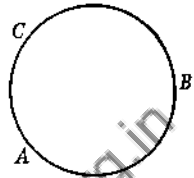
21. Inequalities.—For positive integers it is obvious that if a precedes b and b precedes c, then a precedes c. Hence we have that if $a<b$ and $b<c$, then $a<c$. The number b is between a and c, but not a between b and c.

If $a<b$, $b<c$
then $a<c$.

22. Extension of the fundamental laws.—Up to this point the fundamental laws of arithmetic (and algebra) have been considered for ordinary positive integers. We shall now assume, with no attempts at proof, that these laws hold for all numbers in the real number system. The existence and uniqueness of the result in each of the four fundamental operations are asserted in the equations: $a + b = s$, $a - b = d$, $ab = p$, $a/b = q$, where s, d, p, q, represent sum, difference, product, and quotient respectively. When we write $a = b$, we mean that the different symbols a and b are used to represent the same number. For this reason either one of two symbols connected by the equality sign, =, may be substituted for the other in any algebraic expression, the one to be used depending upon convenience in the particular situation. The equations above are usually called axioms of arithmetic or of algebra and are used as needed in proving other propositions. In this respect they are treated exactly as are the axioms used in the study of geometry.

$a + b = s$	I
$a - b = d$	II
$ab = p$	III
$a/b = q$	IV
$a + b = b + a$	V
$a + b + c = a + (b + c)$	VI
$ab = ba$	VII
$abc = a(bc)$	VIII
$a(b + c) = ab + ac$	IX
If $a<b$, $b<c$, then $a<c$	X
$0 + a = a$	XI
$1 \cdot a = a$	XII

23. Linear and cyclic order.—The properties of inequalities of real numbers rule out what is called cyclic order (order in a cycle). If points A, B, and C lie on a circle, then B is between A and C and C is between B and A, and A is between C and B. In linear order, which we find in real numbers, only one of three numbers can be between the other two. In Chapter 10, when we study another extension of the number system, we shall find numbers that are in neither linear nor cyclic order. The essential property of linear order is stated in X on page 17.



EXERCISES

1. In equations V, . . ., XII, page 17, §22, what are the values of the letters a, b, and c? In equations I, . . ., IV, what are the values of a and b? In which, if any, will the letters a, b, and c have any values whatever, each one entirely independent of the other two?

2. State in words the meaning of each of the equations I, . . ., XII on page 17, §22.

3. In finding the product of 39.76 and 5000, which number would you use as the multiplier? What equation on page 17, §22, permits you to select either number as the multiplier?

4. In checking column addition you may add in the opposite direction. What equations on page 17, §22, state that the sum is the same in whichever order you add? Do you need to use both V and VI? Illustrate by considering $2 + 3 + 4$ and $4 + 3 + 2$.

5. A man earning \$1.25 an hour worked $6\frac{1}{2}$ hours on Monday and $7\frac{1}{2}$ hours on Tuesday. Find his total earnings in two ways. Which way is the easier? What equation on page 17, §22, shows that you are free to use either way?

6. A laborer working at $72\frac{1}{2}$ cents per hour works $7\frac{1}{4}$ hours on Monday, $6\frac{3}{4}$ hours on Tuesday, $7\frac{1}{2}$ hours on Wednesday, $8\frac{1}{4}$ hours on Thursday, $6\frac{1}{2}$ hours on Friday, and $3\frac{3}{4}$ hours on Saturday. What are his total earnings for this week? Find the result to the nearest cent. Note the trouble that is saved by working this the shorter way. What equation on page 17, §22, do you use?

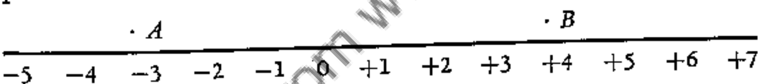
CHAPTER 3:

OPERATIONS ON SIGNED NUMBERS

Before any system of numbers can be put into practical use, we must learn how to perform the fundamental operations on them. When the number system is extended (see §§4, 8, 9), the use and meaning of these operations must be extended to apply to the enlarged system of numbers. In this chapter we shall make a further study of signed numbers and of the fundamental operations on them.

24. The number scale; directed motion; directed segments.—As on page 8, §9, we shall lay off equal units on an endless straight line and label the division points as shown in the figure, positive and negative numbers being indicated by + and − respectively.

The starting point, labeled zero (0), may of course be any point whatever on the line; but when that point is once selected and the unit of measure decided upon, the number corresponding to every point on the line is completely determined.



We shall now add to this the idea of motion in the positive and negative directions along this line, that is, of directed motion. Thus, starting at +1 and going to +5 is a motion of 4 units in the positive direction, while motion from +5 to +1 is a motion of 4 in the negative direction. These motions are designated by +4 and −4 respectively. Similarly a motion from −4 to +2 is represented by +6, while a motion from +3 to −5 is represented by −8.

We also speak of directed segments on this line. Thus the segment *AB* (starting at *A* and ending at *B*) is positive, while the segment *BA* (starting at *B* and ending at *A*) is negative. We say that *BA* is the negative of *AB*. The absolute value of a signed number, written $|a|$ in case the distance is *a*, is the "distance" that it takes us along the number scale irrespective of the direction. Thus the absolute values of −3 and +3 are both 3.

25. ***Addition of signed numbers.***—We shall continue to use the scale of signed numbers:

$-6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6$

To add $+2$ and $+3$, we start at $+2$ and go 3 units in the positive direction. The result is $+5$. To add $+4$ and -3, we start at $+4$ and go 3 units in the negative direction. The result is $+1$. Hence $+4 + (-3) = +1$. To add $+2$ and -5, we start at $+2$ and go 5 in the negative direction; to add -3 and -1, we start at -3 and go 1 in the negative direction; to add -2 and $+5$, we start at -2 and go 5 in the positive direction; to add $+6$ and -6, start at $+6$ and go 6 in the negative direction. Thus the results given at the right are easily verified on the number scale.

$$+2 + (+3) = +5 \qquad (1)$$
$$+4 + (-3) = +1 \qquad (2)$$
$$+2 + (-5) = -3 \qquad (3)$$
$$-3 + (-1) = -4 \qquad (4)$$
$$-2 + (+5) = +3 \qquad (5)$$
$$+6 + (-6) = 0 \qquad (6)$$

It is evident that adding $+a$ and $-a$ (a itself being positive) gives zero, since this represents a motion from zero a distance a in the positive direction and then a distance a in the negative direction, which of course takes us back to the starting point.

We shall now assume that for every number $+a$ there is a number $-a$ having the opposite sign such that $+a + (-a) = 0$.

$$+a + (-a) = 0$$

It is easily seen that the additions of signed numbers shown above are covered by the following rules.

1. *To add numbers having the same sign, add their absolute values and prefix the common sign.* This covers (1) and (4) above.

2. *To add two numbers having opposite signs, find the difference between their absolute values and prefix the sign of that one whose absolute value is the greater.* This covers (2), (3), and (5) above.

3. *The sum of two numbers with opposite signs and the same absolute value is zero.* This covers (6) above.

Check each of the six examples given above to see that the result follows from one of these three rules.

The equation $a + (-a) = 0$ may also be regarded as the definition of the negative number in that it gives the essential relation between a and $-a$.

The sum of numbers obtained according to this rule is called the algebraic sum of the numbers. The essential property of positive and negative numbers with respect to addition is that each kind of number tends to destroy or "cancel" the other, exactly as do motions in opposite directions on a straight line. The sum of a number of such motions is regarded as the distance and direction of the line from the starting point to the final stopping point. The algebraic sum of the numbers representing these motions gives the length and direction of this line.

26. ***Subtraction of signed numbers.***—The subtractions shown here are easily verified by referring to the scale of signed numbers.

That is, in $+6-(+4)$, we start at $+4$ and go 2 in the positive direction in order to reach $+6$; in $+6-(+8)$, we start at $+8$ and go 2 in the negative direction to reach $+6$; and so on for the others. All these results are obtained by using the following rule.

$+6-(+4)=+2$	$-6-(-4)=-2$
$+6+(-4)=+2$	$-6+(+4)=-2$
$+6-(+8)=-2$	$-4-(-6)=\ \ 2$
$+6+(-8)=-2$	$-4+(+6)=\ \ 2$
$+6-(-4)=\ 10$	$-6-(-6)=\ \ 0$
$+6+(+4)=\ 10$	$-6+(+6)=\ \ 0$

In subtraction change the sign of the subtrahend and add the result to the minuend.

The adding must of course be done according to the rules for adding signed numbers.

Check the examples given above, (a) to see that the results follow from the definition of subtraction, (b) to see that they follow from the rule for subtraction just stated.

27. ***The negative of a number.***—If we have $a+b=0$, both a and b being different from zero, then we know that one of these is negative and the other is positive. We may then write $a=-b$ and also $b=-a$. That is, if $a+b=0$, then a is the negative of b and b is the negative of a. To take the negative of a number, simply change its sign. Thus -2 is the negative of 2, and 2 is the negative of -2.

Changing the sign of a number may be regarded as changing the direction in which it takes us along the line, the distance which it takes us remaining unchanged.

28. *Multiplying signed numbers.*—Starting with a and b as positive numbers and the general idea that positive numbers are identical with the numbers of arithmetic, we have (1) at the right.

$$(+a)\cdot(+b) = +(ab) \quad (1)$$
$$(+a)\cdot(-b) = -(ab) \quad (2)$$
$$(-a)\cdot(+b) = -(ab) \quad (3)$$
$$(-a)\cdot(-b) = +(ab) \quad (4)$$

Since $-b$ extends a distance b in the negative direction, $+a(-b)$ is regarded as a operating on b in that direction forming an absolute magnitude ab, but negative in quality, giving $-(ab)$. Since $(-a)\cdot(+b) = (+b)\cdot(-a)$ by the commutative law of factors, we have $(-a)\cdot(+b) = -(ab)$. This gives (2) and (3) as stated above.

In $(-a)\cdot(-b)$, we regard the operation by $-a$ as reversing the direction of $-b$. Hence the result is the same as using $+a$ as the multiplier and $+b$ as the multiplicand. This leads to (4) above. Hence we have the following rules.

1. *If two factors have the same sign, the product is positive.*
2. *If two factors have opposite signs, the product is negative.*
3. *The absolute value of the product is the product of the absolute values of the factors.*

29. *Division of signed numbers.*—From the definition of division, $\frac{a}{b}\cdot b = a$, the statements at the right are easily verified. The rule is:

$\frac{+a}{+b}$ is positive

$\frac{-a}{-b}$ is positive

$\frac{-a}{+b}$ is negative

$\frac{+a}{-b}$ is negative

1. *If the dividend and divisor have the same sign, the quotient is positive.*
2. *If the dividend and divisor have opposite signs, the quotient is negative.*
3. *The absolute value of the quotient is the quotient of the absolute values of the dividend and divisor.*

By noting that $\frac{a}{b} = a\cdot\frac{1}{b}$ and that $-\frac{1}{b} = \frac{1}{-b}$, the rule of signs in multiplication is seen to apply directly to fractions.

That is, $\frac{-a}{-b} = -a\cdot\frac{1}{-b} = -a\left(-\frac{1}{b}\right) = a\cdot\frac{1}{b} = \frac{a}{b}$; and similarly in the last two cases.

30. Rule of signs in products and quotients.—The following is now obvious. If one factor in a product is negative and the other factors are positive, then the product is negative. If two factors are negative and the others are positive, the product is positive; if three factors are negative, the product is negative, and so on. In the equations at the right a, b, c, d, e, p, and q are all positive. We then easily verify that:

$$a \cdot b \cdot c \cdot d \cdot e = p$$
$$a \cdot b \cdot c \cdot d(-e) = -p$$
$$a \cdot b \cdot c(-d)(-e) = p$$
$$a \cdot b(-c)(-d)(-e) = -p$$
$$a(-b)(-c)(-d)(-e) = p$$
$$(-a)(-b)(-c)(-d)(-e) = -p$$
$$\frac{a \cdot b \cdot c}{d \cdot e} = q$$
$$\frac{a \cdot b(-c)}{d \cdot e} = -q$$
$$\frac{a(-b)c}{(-d)e} = q$$
$$\frac{(-a)b(-c)}{d(-e)} = -q$$

1. A product is positive if an even number (including zero as an even number) of its factors are negative, the remaining factors being positive. If an odd number of factors are negative, the product is negative.

2. A quotient is positive if an even number of factors in the dividend and divisor taken together are negative. Otherwise the quotient is negative.

It follows that the sign of a product or a quotient remains unchanged when the signs of an even number of factors are changed, but that the sign of a product or a quotient is changed when the signs of an odd number of factors are changed.

EXERCISES

1. Find the product in each of the following. $2 \cdot 3 \cdot 4 \cdot 5$, $6 \cdot 2 \cdot (-5)$, $(-4)(-5)(-6)$, $(-2) \cdot 3 \cdot 4(-5)(-6)$, $-(-2)(-3)(-4) \cdot 5 \cdot 6$, $(-2)(-3)(-4)(-5)(-6)$.

2. Find the quotient in each of the following.

$$\frac{12 \cdot 15 \cdot 27}{9 \cdot 6 \cdot 5}, \quad \frac{12(-15) \cdot 27}{-9 \cdot 6 \cdot 5}, \quad \frac{-12 \cdot 15(-27)}{9(-6) \cdot 5}, \quad -\frac{(-12)(-15)(-27)}{(-9) \cdot 6 \cdot 5}$$

3. Are any two of the following equal to each other?

$$\frac{(a-b)(b-c)(c-d)}{(p-q)(q-r)(r-s)}, \quad -\frac{(b-a)(c-b)(c-d)}{(p-q)(r-q)(r-s)}, \quad \frac{(a-b)(c-b)(c-d)}{(q-p)(r-q)(s-r)}$$

31. Examples in the use of signed numbers.—Clearly there are many situations to which ordinary numbers of arithmetic apply, but to which negative numbers do not apply. Thus it does not make sense to say that there are negative four books on a shelf, while it does make sense to say that the temperature is negative four degrees or that in a business transaction the "gain" is a negative quantity. Such a difference is equally clear as between integers and fractions. One may look out of a window twice but not 2½ times, or there may be 7 elephants in the circus parade but not 7 4/5 elephants.

The service rendered to mathematics by signed numbers lies in the comparatively great simplicity that they make possible. This is illustrated in the Examples below. In the more general uses of numbers in further work in mathematics and in the application of mathematics in the sciences and in engineering, this simplification is so great that without it much of the work that is done would be practically impossible.

It is not altogether satisfactory to use the signs + and − with two distinct meanings for each, indicating as they do addition and subtraction, and also the positive and negative quality of numbers. But this usage is now established and no really great difficulty results from it.

Example 1. The average of the n numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

If these n numbers represent temperature readings, for example, as may easily be the case, then some or all of them may be positive and the rest negative, or all may be negative.

Note that this formula, exactly as it stands, is a complete rule for finding the average of any possible combination of readings. To evaluate this formula in any given case, it is first necessary to find the sum of $a_1 + a_2 + \ldots + a_n$, which involves the addition of signed numbers. The sum may be positive or negative, and this signed number must be divided by n.

To begin to appreciate the usefulness of signed numbers in this case it is worth while to write in words a complete rule for finding the average of a set of temperature readings that will apply to all possible cases. Write such a rule.

Example 2. A train going r miles per hour on an east and west road passes through a certain town exactly at noon. Find the location of the train at a certain time t.

SOLUTION. Let d be the distance from the town at the time t. Then $d = rt$ will give the distance from the town provided t is measured from twelve o'clock. Let time after twelve noon be positive and time before noon negative. Let rates of motion and also distances eastward be positive, and westward negative.

$$d = rt$$

Let r have an absolute value of 40 miles per hour. Under these conditions at 3 P.M. ($t = +3$) the train will be 120 miles east of the station if it is running eastward. ($r = +40$, $t = +3$)

Formula: $d = tr$	Direction of motion
$d = (+3) \cdot (+40) = +120$	eastward
$d = (-3) \cdot (+40) = -120$	eastward
$d = (+3) \cdot (-40) = -120$	westward
$d = (-3) \cdot (-40) = +120$	westward

At 9 A.M. ($t = -3$) this train was 120 miles west of the station. ($r = +40$, $t = -3$)

At 3 P.M., train going westward, $t = +3$, $r = -40$.

At 9 A.M., train going westward, $t = -3$, $r = -40$.

EXERCISES

1. Explain fully the meaning of each of the four equations given above.
2. Find the average of each of the following sets of temperature readings.

(a) 14°, 16°, 20°, 20°, 18°, 15°, 13°, 10°, 9°, 7°, 5°, 3°.
(b) 16°, 14°, 8°, 6°, 6°, 4°, 0°, −2°, −4°, −6°, −4°, −8°.
(c) 8°, 4°, 2°, 2°, −1°, −3°, −6°, −7°, −9°, −10°, −12°, 12°.
(d) −1°, −2°, −5°, −6°, −4°, −5°, −8°, −10°, −9°, −9°, −8°.

3. In Example 2 above, let the speed of the train be 35 ($|r| = 35$), the remaining arrangements being the same as in that Example. Find the values of d, including sign, for 5 P.M. and 8 A.M., the train going eastward; also for 4 P.M. and 9 A.M., the train going westward.

CHAPTER 4:

DEFINITIONS; OPERATIONS ON ALGEBRAIC EXPRESSIONS

While the fundamental operations are in general principle the same in algebra as in arithmetic, literal expressions frequently cannot be consolidated into a single term; and this makes the carrying out of these operations, in appearance at least, quite different from what they are in arithmetic. The purpose of this chapter is to study these operations as they must be carried out in algebra. We shall begin with a collection of definitions, which are grouped here for convenient reference, though some of the terms defined have been used informally earlier.

32. ***Algebraic expressions.***—Any expression used in algebra, consisting of ordinary numbers and letters representing numbers, for the purpose of representing quantities is called an algebraic expression. The first part of this chapter consists of a study of the symbols used in such expressions and the actual building up of the more common algebraic expressions.

33. ***Parentheses.***—Ordinary parentheses, ( ); braces, { }; brackets, [ ]; and upper bars, ——, are used as symbols of aggregation. We shall refer to all of these as parentheses. The reason that several symbols of aggregation are used is that it may be necessary to use one within another.

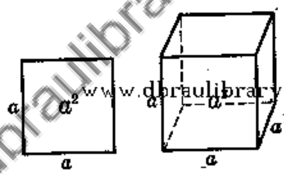
If an expression is enclosed in parentheses, this means that any symbol of operation used in connection with the parenthesis is to affect the whole expression. Thus $a - (b + c)$, $a - \{b + c\}$, $a - [b + c]$, $a - \overline{b + c}$ all mean that the sum of b and c is to be subtracted from a. That is, the $-$ sign before the parenthesis affects both the b and the c. Again $a(b + c)$, or $(b + c)a$ means the product of a and the sum of b and c, while $a \div (b + c)$ means that a is to be divided by the sum of b and c. That is, $a \div (b + c) = \frac{a}{b + c}$. Also $(a + b)^2$ means that $a + b$ is to be multiplied by itself.

34. Exponents.—A positive integer written above and a little to the right of an expression is called an exponent, and shows how many times that expression is to be used as a factor. The meaning of other numbers used as exponents will be studied later.

$$a^2 = a \cdot a$$
$$a^3 = a \cdot a \cdot a$$
$$a^4 = a \cdot a \cdot a \cdot a$$

35. Power; base.—An expression consisting of a number affected with an exponent is called a power. The expression affected with the exponent is the base of the power. Thus a^3 is a power of a, and a is the base of this power.

The expressions a^2, a^3 may be used to represent respectively the area of a square with sides a and the volume of a cube whose sides are a. For this reason a^2 and a^3 are called "a square" and "a cube" respectively. Expressions such as a^4 and a^5 are referred to as "the fourth power of a," "the fifth power of a," and also as "a exponent 4" and "a exponent 5." An exponent affects only the factor next to which it is written. Thus ab^2 means that the square of b is to be multiplied by a, while $(ab)^2$ means the square of the product ab, and $(a + b)^3$ means the cube of the sum of a and b. The expression a^1 means the same as a. In this case the exponent is usually omitted.



EXERCISES

1. For $a = 2$, $b = 3$, $c = 4$, find the values of abc^2, ab^2, a^2bc, $a(bc)^2$, $(ab)^2c$.
2. For $a = \frac{1}{2}$, $b = 2$, $c = \frac{1}{2}$, find the values of a^2bc, a^2b^2c, ab^2c^2, ab^2c. (Note that $(\frac{1}{2})^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.)
3. For $a = 2$, $b = 3$, $c = 4$, find the values of a^3bc, ab^3c, abc^3, a^2bc^3, a^2b^3c, $a^3b^2c^2$.
4. For $a = 2$, $b = 3$, $c = 4$, find the values of $a + b - c$, $a + (c - b)$, $a - (c - b)$, $(-a + b) + c$.
5. For $a = 2$, $b = 3$, $c = 4$, find the values of $a(b + c)$, $a(c - b)$, $-(a + b - c)$, $-(a + c - b)$.
6. For $a = 2$, $b = 3$, $c = 4$, find the values of $(-a)^2(a + b)$, $(-a)^3b(-c)$, $ab(-c)^2$, $a(-b)^2(-c)$, $(-a)^2b^2(-c)$, $(a + b)^2(-c)$, $a^2(b - c)^3$.
7. For $a = -2$, $b = -3$, $c = -4$, find the values of $(-a)^2(a + b)$, $(-a)(a^2 - b)$, a^3b^2c, $(a - b)(b - c)$, $(c - a)(b - c)$.

36. *Factors; coefficients.*—If two or more numbers are multiplied to form a product, then these numbers are factors of the product. In an indicated product any one of the factors may be called the coefficient of the product of the other factors. Thus in $2xy$, 2 is the coefficient of xy, x is the coefficient of $2y$, and y is the coefficient of $2x$. A factor expressed as an ordinary numeral is called a numerical coefficient. In an expression such as ax^2, a is often spoken of as the coefficient (the literal coefficient) of x^2, while only in exceptional circumstances is x regarded as the coefficient of ax in ax^2.

37. *Term; monomial; binomial; trinomial; polynomial.*—An indicated product or quotient of ordinary numbers or literal numbers is called a term or monomial. The sum (or difference) of two monomials is a binomial and the sum of three monomials is called a trinomial. Any expression consisting of two or more terms is called a polynomial, though "binomial" is usually used in the case of two terms. In a polynomial, the terms are connected by the + or − signs.

In a monomial one or more of the factors may be a polynomial. Thus $a(b + c)$, $(a + b)(c - d)$ are really monomials. Parentheses may be placed about a polynomial to make it a monomial.

38. *Similar terms.*—If two terms have a common factor, they are said to be similar with respect to that factor. Thus ax and bx are similar with respect to x. The terms $4xy^2$ and $6xy^2$ are similar with respect to xy^2 and also with respect to x, y, and y^2. Similar terms are also called like terms.

Sometimes terms are said to be similar only in case their literal factors are the same. By this usage $12ab$ and $3ab$ are similar, but not ac and bc. We shall adopt the more general meaning of this word as stated above.

At times we shall also say that terms are similar with respect to a fractional factor. Thus the terms at the right may be said to be similar with respect to $\frac{n(n-1)}{2}$, the coefficients of this common factor being $\frac{n-2}{3}$ and 1.

$$\frac{n(n-1)(n-2)}{2 \cdot 3}, \quad \frac{n(n-1)}{2}$$

39. *Square roots; cube roots.*—One of the two equal factors of a number is a square root of the number, while one of the three equal factors of a number is a cube root of the number. The radical sign, $\sqrt{\ }$, is used to indicate a root. Thus $\sqrt{4}$ is a square root of 4 and $\sqrt[3]{8}$ is a cube root of 8. In general $\sqrt[m]{a}$ is an mth root of a. In this symbol, m is called the index of the root and a is called the radicand. While 4 has two square roots, namely 2 and -2, $\sqrt{4}$ indicates only one of these, namely 2, while -2 as a root may be indicated by $-\sqrt{4}$.

Every positive number has two real square roots which may or may not be rational. One of these roots is positive and one negative. Thus 4 and 9 have two rational square roots each, while the square roots of 2 are irrational. A negative number has no real square roots. Every real number, positive or negative, has one real cube root. In general every positive number has two real even roots, and a negative number has no real even roots. Every real number has one real odd root.

40. *Rational integral expressions; rational fractions.*—The expressions $ax + b$, $ax^2 + bx + c$, . . . , $ax^n + bx^{n-1} + \ldots + bx + k$ (n integral) are said to be rational and integral in x, whatever kinds of numbers may be represented by the coefficients $a, b, c, \ldots$ That is, $\frac{3x-7}{14}$, $\frac{2x^2 - \sqrt{3}x + \sqrt[3]{4}}{4}$, are rational integral expressions in x. These may be written $\frac{3}{14}x - \frac{1}{2}$ and $\frac{1}{2}x^2 - \frac{\sqrt{3}}{4}x + \frac{\sqrt[3]{4}}{4}$.

The expression m/n, where m and n are integers ($n \neq 0$), is a rational fraction. The general expression at the right is a rational, but not an integral, expression in x for all values of $a, b, c, \ldots, p, q, r, \ldots$, not all of $p, q, r, \ldots$ being zero, provided m and n are integers.

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \cdots}{px^n + qx^{n-1} + rx^{n-2} + \cdots}$$

EXERCISES

1. If A and B are rational integral expressions in x, which of the expressions $A + B$, $A - B$, AB, A/B will be rational integral expressions in x?

2. If A and B are rational (not necessarily integral) expressions in x, which of the expressions $A + B$, $A - B$, AB, A/B will be rational expressions in x? Distinguish clearly between examples 1 and 2.

41. ***Polynomials in x.***—The word polynomial is sometimes restricted to refer to a special kind. The expression at the right is said to be a polynomial in x for all values of $a, b, c, \ldots,$ provided n is a positive integer.

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + bx + k$$

If a is not zero, this polynomial is of degree n. Any of the other coefficients may or may not be zero. Thus $x^4 - 7x^3 - 3x^2 + 2x - 6$, $x^4 - 8x^2 + 3$, and $x^4 - 1$ are all fourth-degree polynomials in x; and $x^n \pm 1$ are polynomials in x of degree n.

42. ***Adding and subtracting monomials.***—The following rules may be used.

To add similar terms add coefficients of the common factor and multiply the sum by the common factor.

To subtract similar terms multiply the difference of the coefficients by the common factor.

These rules are immediate consequences of the distributive law of multiplication with respect to addition and subtraction.

$$ax + bx = (a + b)x$$
$$ax - bx = (a - b)x$$

Thus $3x + ax - bx = (3 + a - b)x$, $abx^2 + 3bx^2 - cbx^2 = (a + 3 - c)bx^2$, and $\frac{n(n-1)(n-2)}{2 \cdot 3}xy + \frac{n(n-1)}{2}xy = \left[\frac{n-2}{3} + 1\right]\frac{n(n-1)}{2}xy = \frac{n+1}{3} \cdot \frac{n(n-1)}{2}xy$. This operation may also be regarded as a case in factoring. In fact this is usually done in more advanced work. We "take out" the common factor.

43. ***Adding and subtracting polynomials.***—In adding or subtracting polynomials, combine similar terms. It may be convenient to arrange such terms in columns.

Example 1. At the right is shown the addition of the polynomials $4x^3 - 6x + 7$, $x^4 - 5x^2 + 2x - 3$, and $x^3 - 9x^2 + 7x - 2$.

$$\begin{array}{rrrrr} & 4x^3 & & -\ 6x & +\ 7 \\ x^4 & & -\ 5x^2 & +\ 2x & -\ 3 \\ & x^3 & -\ 9x^2 & +\ 7x & -\ 2 \\ \hline x^4 & +\ 5x^3 & -\ 14x^2 & +\ 3x & +\ 2 \end{array}$$

Study the arrangement of this work. Note the spaces left in writing the first and second polynomials. Such sums may often be read without writing the addends again. Thus looking along these addends we see that x^4 is the highest power of x and that there is only one such term. Then we see $4x^3$ and x^3, and write the sum $5x^3$, and so on.

Example 2. From $7x^5 - 4x^3 + 2x^2 - 9$ subtract $5x^4 + 2x^3 - 3x^2 + 2x - 12$. Study the arrangement and the work at the right.

$$\begin{array}{rrrrrr} 7x^5 & & -\,4x^3 & +\,2x^2 & & -\ \ 9 \\ & 5x^4 & +\,2x^3 & -\,3x^2 & +\,2x & -\,12 \\ \hline 7x^5 & -\,5x^4 & -\,6x^3 & +\,5x^2 & -\,2x & +\ \ 3 \end{array}$$

EXERCISES

Add:

1. $\begin{array}{r} 4ax^3 \\ 7ax^3 \\ -4ax^3 \\ cax^3 \\ \hline \end{array}$

2. $\begin{array}{r} 3b^2c \\ b^2c \\ ab^2c \\ -2b^2c \\ \hline \end{array}$

3. $\begin{array}{r} na^3b \\ \frac{n}{2}a^3b \\ \frac{n(n+1)}{2}a^3b \\ \hline \end{array}$

4. $\begin{array}{r} (a+b)m^2n \\ (a-b)m^2n \\ (a+2b)m^2n \\ (2a-b)m^2n \\ \hline \end{array}$

Subtract:

5. $\begin{array}{r} ab^3x^2 \\ 3b^3x^2 \\ \hline \end{array}$

6. $\begin{array}{r} cm^2n^2 \\ 4m^2n^2 \\ \hline \end{array}$

7. $\begin{array}{r} na^3b^2 \\ (n-1)a^3b^2 \\ \hline \end{array}$

8. $\begin{array}{r} (m+n)pq \\ (m-n)pq \\ \hline \end{array}$

9. Add $7x^3 - 2x + 2$, $5x^2 + 8x$, $3x^3 + 5x^2 - 9$.

10. Add $2ax^3 - bx^2 + cx - d$ and $6x^3 + bx^2 - 5x + d$.

11. Add $5ab^3 - 3a^2b + 2a^2b^2$ and $2ab^3 + 5a^2b + 5a^2b^2$.

12. From the sum of $7x^2 - 9x + 4$ and $3x^3 - 2x^2 + 12x - 2$ subtract $2x^3 - 9x^2 + x - 1$.

13. From $9x^4 - 2x^2 + 3x - 9$ subtract the sum of $2x^3 - x^2 + 3x + 2$ and $4x^4 + 5x^2 - 9x + 4$.

14. From the sum of $3x^5 + 7x^2 - 2x - 3$ and $5x^4 - 6x^2 + 12$ subtract the sum of $3x^4 - 4x^3 + 8x - 3$ and $x^5 - 2x^4 - 9x^3 + 2x - 1$.

15. From the difference of $2x^3 - 9x^2 - 2x + 8$ and $4x^4 - 7x + 9$ subtract the difference of $2x^4 - 9$ and $3x^3 - 7x^2 - x + 8$. (In each case let the "difference" be the first expression less the second.)

16. Subtract $5a^3 + 8a^2 - 7a - 4$ from $8a^4 - 3a^3 + 2a - 6$ and then subtract $2a^3 - 7a^2 + 2a - 5$. Check by finding the final result in two different ways.

Check the following addition.

17. $\frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}ab + \frac{n(n-1)(n-2)}{2 \cdot 3}ab = \left[\frac{n-3}{4} + 1\right] \frac{n(n-1)(n-2)}{2 \cdot 3}ab = \frac{n+1}{4} \cdot \frac{n(n-1)(n-2)}{2 \cdot 3}ab = \frac{(n+1)n(n-1)(n-2)}{2 \cdot 3 \cdot 4}ab.$

44. Removing parentheses.—From the meaning of parentheses (page 26, §33), it follows that a parenthesis with a plus sign before it indicates that the expression within is to be added, while a minus sign indicates that the expression is to be subtracted. It follows at once that a plus sign before a parenthesis and also the parenthesis may be removed without in any way changing the expression within. It also follows that a minus sign before a parenthesis and also the parenthesis may be removed by changing the sign of every term within.

Thus,

$$7a + 3b + (2a - 5b + 3) = 7a + 3b + 2a - 5b + 3 = 9a - 2b + 3$$

In this case the + sign, understood before the $2a$ within the parenthesis, is supplied when the parenthesis is removed.

Again,

$$8x^2 + 5x - 7 - (2x^2 - 4x + 2) = 8x^2 + 5x - 7 - 2x^2 + 4x - 2 \\ = 6x^2 + 9x - 9$$

If there is a factor to be multiplied into the expression in the parenthesis, this multiplication should be carried out as the parenthesis is removed. For this multiplication see page 36, §48.

Thus,

$$5m^2 - 3m + 2 + 2(m^2 - 3m + 4) = \\ 5m^2 - 3m + 2 + 2m^2 - 6m + 8 = 7m^2 - 9m + 10$$

and

$$2x^3 - 6x + 3 - 4(x^2 + 4x - 9) = \\ 2x^3 - 6x + 3 - 4x^2 - 16x + 36 = 2x^3 - 4x^2 - 22x + 39$$

If one parenthesis occurs within another, then these should be removed in order, beginning with the one that is inside.

Thus,

$$15 - \{2x + 3 - (5x - 1) + 4\} = \\ 15 - \{2x + 3 - 5x + 1 + 4\} = 15 - \{-3x + 8\} \\ = 15 + 3x - 8 = 3x + 7$$

Note that we simplify the expression with the { } before removing this pair. This step may equally well be taken in the other order.

EXERCISES

Using the principles in §42, perform the indicated operations in the following.

1. $17(x + 5) + c(x + 5) + 5b(x + 5)$
2. $3(x + y - z) + (z - 2b + 3c)(x + y - z)$
3. $5b(m^2 - n^2) - 3c(m^2 - n^2) + (a + 3)(m^2 - n^2)$
4. $(a + b)(x^2 + y^2) + 3a(x^2 + y^2) - (2a + 3b)(x^2 + y^2)$
5. $(2x + 3y)(a - b) - (2x + 3y) + (a + b)(2x + 3y)$
6. $(a^3 + b^3) + c(a^3 + b^3) - 5c(a^3 + b^3)$

7. $3c(m^2 + mn + n^2) + a(m^2 + mn + n^2) + b(m^2 + mn + n^2)$
8. $7x(x + a - c) + 6y(x + a - c) + (3 - 5y)(x + a - c)$
9. $5c(x - 1) - 3a(x - 1) + 5b(x - 1)$
10. $3r(9x + 5) + 8n(9x + 5) + 4m(9x + 5)$
11. $11y(a + 2b - 3c) - 5z(a + 2b - 3c) + 7x(a + 2b - 3c)$
12. $13a(m - 3n) + 7b(m - 3n) - 15c(m - 3n)$

13. $a(r - s + 2) - b(r - s + 2) + 2(r - s + 2)$
14. $(a - b)(2c - 3d) + 2a(2c - 3d) + b(2c - 3d)$
15. $a(m^2 - n^2 - 3) - b(m^2 - n^2 - 3) + 2(m^2 - n^2 - 3)$
16. $(r - s + 2) - 4(r - s + 2) + 7(r - s + 2)$
17. $(a + b + c)(x + y) - a(x + y) - b(x + y) + (x + y)$
18. $n(n - 1)(n - 2) - n^2(n - 2) + (n - 2)$

19. $cd(2a - 4 + b) - c(2a - 4 + b) - c(2a - 4 + b)$
20. $(3a - 2b)(a + b) + 2b(a + b) - 2a(a + b)$
21. $(5x + 3y)(m^2 - n) - 7y(m^2 - n) - 3x(m^2 - n)$
22. $(2 - 3a)(m - n - p) + 2a(m - n - p) - (m - n - p)$
23. $(5r + 3)(2a - 3b + c) + (3r - 2)(2a - 3b + c) - (2a - 3b + c)$
24. $(m^2 - mn + n^2)(3a + 4) - 5b(m^2 - mn + n^2) - (2c + 4)(m^2 - mn + n^2)$

Remove parentheses as in §44 and then collect similar terms in the results.

25. $3[4x + 3 + 2\{5x - 7\} - 2x + 6]$
26. $18a - \{2 - 6a - 6(2a + 4) + 16a\} + 8a$
27. $5 + 4(b + c + 2\{b - c\} - 5a - 2b)$
28. $4xy - 3[5x - 2 + x(4 - y) + xy]$
29. $6pq^2 + p(q + 2) - [6pq - 3p] + 5pq^2$
30. $37rs - r(r + s) - 5\{7r - 3s\} - 15rs$

45. Laws of exponents.—For the present, let exponents take only positive integral values. Then a^n is simply a short way of indicating that a is used n times as a factor. That is, $a \cdot a \cdot a \cdot \ldots$ (to n factors a) $= a^n$.

$$\begin{aligned} a^m \cdot a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \\ a^m \div a^n &= a^{m-n} \\ a^0 &= 1 \end{aligned}$$

In $a^m \cdot a^n$ we have a used as a factor $m + n$ times. That is, in a^n, a is already used n times. Starting to multiply this by a^m, multiply by a, obtaining $n + 1$ factors, or a^{n+1}; then by a again, obtaining $n + 2$ factors, or a^{n+2}; and so on until we have used up the m factors a in a^m, when we have $m + n$ factors in the product.

Hence we have the rule: *To multiply powers of the same base, leave the base unchanged and add the exponents.*

In $(a^m)^n$, a^m is used as a factor n times, and hence a is used mn times. That is, $(a^m)^n = a^{mn}$.

That $a^m \div a^n = a^{m-n}$ follows directly from the definition of division since $a^n \cdot a^{m-n} = a^{m-n+n} = a^m$, provided for the present that n is not greater than m. Hence we have the rule:

To divide powers of the same base, leave the base unchanged and subtract the exponents.

If we are to give the zero exponent a meaning that preserves the law of adding exponents, then $a^0 = 1$, since $a^n \cdot a^0 = a^{n+0} = a^n$, and hence $a^0 = a^n \div a^n = 1$.

46. Multiplying monomials.—*To multiply an indicated product by a number, multiply any one of the factors by that number.* Thus $3acx$ multiplied by c equals $3ac^2x$. This follows directly from the commutative and associative laws of factors. That is,

$$c \cdot 3acx = 3ac^2x$$

$$\begin{aligned} c \cdot 3acx &= 3accx && \text{(the commutative law)} \\ 3accx &= 3ac^2x && \text{(the associative law)} \end{aligned}$$

To multiply by a product, multiply by each of the indicated factors of the product. Thus, to multiply by $2ax^2$ multiply by x^2, a, and 2. In the example at the right we multiply the x^2 into the factor x, a into the factor a^2, and 2 into the factor 6.

$$2ax^2 \cdot 6a^2xy = 12a^3x^3y$$

The following rule may be used.

To find the product of two terms, multiply the numerical coefficients and add the exponents of each letter in the multiplier to the exponent of the same letter in the multiplicand.

If, as in the above case, a letter occurs in one of the terms to be multiplied and not in the other, we may regard the exponent as zero in that term and the rule holds exactly as written.

Each letter must be used as many times as a factor in the product as it is in both the multiplier and the multiplicand.

47. ***Dividing monomials.***—It follows directly from the definition of division that the following rule holds.

To find the quotient of two monomials, divide the numerical coefficients and subtract exponents of like bases.

Since this at times leads to negative exponents, we may indicate the division as a fraction and then remove like factors from the numerator and the denominator, leaving the result in the fractional form.

EXERCISES

In each of the following multiply the first term by the second. Then divide the first term by the second.

1. $6a^2b,\ 3ab^2$
2. $-bc^2d,\ 7ab$
3. $4a^2b,\ 2a^3c$
4. $2x^2y^2,\ -x^3y^3$
5. $5m^2n^4,\ 10mn^3$
6. $mn^3r^3,\ -2m^2n^2r^2$
7. $a^2b^2,\ ab^4$
8. $3bc^2,\ 4b^2d^2$
9. $-a^2x^3,\ -8a^4x^2$
10. $3x^4y^3,\ 5yz^4$
11. $2r^3s^5,\ 6r^2s^2$
12. $y^4z^4,\ -y^3z^3$
13. $2a^5x^7,\ 7a^6y^3$
14. $-a^3x^5,\ -9a^3x^3$
15. $5y^4z^3,\ 2yz^4$
16. $4a^5x,\ 4xy^3$
17. $3b^4x^2,\ 2bx^3$
18. $b^2c^3,\ -a^2b^3c^2$

Write the products of the given terms in each of the following.

19. $3ab,\ 2a^3bc,\ 4ac^3,\ 2bc$
20. $a,\ b^3,\ c^2,\ 2a,\ 3abc$
21. $2x,\ -3y,\ 2a,\ -x^2,\ xy$
22. $5m,\ 2mn,\ -3p^2,\ -2mnp$
23. $3p,\ -4q^2,\ -p^2,\ 2m^2$
24. $-x^3,\ 4x,\ -2xy,\ -3xy^2$
25. $7r,\ -2s,\ -2rs,\ -r^3s^2$
26. $5l^2,\ -2m^2,\ -3n^2,\ -p^2$
27. $4a,\ -b^2,\ -c^3,\ -3abc$
28. $2q,\ -2p,\ -2r,\ -2s$
29. $4c,\ 2a,\ -3ac,\ 2bc$
30. $-6a,\ 6c,\ -ab,\ -ac$

Find the result in each of the following indicated operations.

31. $(3x^2y \cdot 2yz^2) \div 6xyz$
32. $(a^2bc \cdot 4b^3c^2) \div 2ab^2c^5$
33. $(2pq \cdot 8p^2q^3) \div 4p^4q^3$
34. $(-r^2s^2 \cdot 7pr^2) \div (-3p^3r^2s)$
35. $(6a^3b^2c \cdot 2bc^4) \div (-2a^3bd)$
36. $[2x^2y \cdot (-4xy^4)] \div (-6x^3y^2)$
37. $-3p^2q^3 \cdot (-6pr^2s) \div (-9rs^2p^4q^6)$
38. $7mnx \cdot (-4mx^2z) \div (-14mn^3x^2z^2q)$

48. ***Multiplying a polynomial by a monomial.***—It is a direct consequence of the distributive law of multiplication that:

To multiply a polynomial by a monomial, multiply each term separately and collect the products.

49. ***Dividing a polynomial by a monomial.***—From the definition of division as the inverse of multiplication we have:

To divide a polynomial by a monomial, divide each term separately and collect the quotients.

In case the division does not lead to integral quotients, leave the quotients in fractional form as indicated at the right.

$$\frac{4x^3 - 6x^2 + 7x - 4}{2x} = 2x^2 - 3x + \frac{7}{2} - \frac{2}{x}$$

Note in particular that to multiply or divide a product by a number, we multiply or divide only one of the factors; but to multiply or divide a sum or a difference we multiply or divide *each term separately*.

EXERCISES

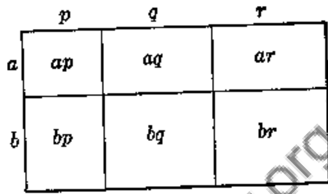
Carry out the following indicated multiplications and divisions.

1. $5x(3x^3 - 5y + 6)$
2. $2a^2b(ab^2 + 4ab^3)$
3. $(14x^2 - 3x + 7) \cdot 5x$
4. $3am(2a^3 - 3m^2 + 1)$
5. $(7rs - 3r + 5s) \cdot 5as$
6. $12ax^2(2ax - 4a^2x + 3ax^3)$
7. $32(2a^2 - 3b^2 + ab)$
8. $5rs(3r^2 + 2s^2 - 6rs)$
9. $3a^2(5ab - 7b^2c - 3c^2)$
10. $a^3(a + 1 + 7b - 8c)$
11. $2a^2r^2s^2(14ar + 7rs - 7r^2s^2)$
12. $(10pq - 3p^2 - 2q^2)6pr$
13. $\dfrac{8x^5 - 4x^3 + 2x^2}{2x^2}$
14. $\dfrac{5a^2b - 5ab^2 + 2ab}{ab}$
15. $\dfrac{4p^2q^2 - 8pq + 6}{2pq}$
16. $\dfrac{6mn - 8m^2n + 10mn^2}{2mn}$
17. $\dfrac{16ar^2 + 8a^2r - 12}{4ar}$
18. $\dfrac{12pqr - 6pq + 3qr}{3pq}$
19. $\dfrac{24m^3 - 18n^2 + 12mn}{6mn}$
20. $\dfrac{x^4 - 8x^3 + 6x^2 + 4x - 12}{4x^2}$
21. $\dfrac{3a(6ab^2 - 3a^2 - 3b^2)}{6a^2b^2}$
22. $\dfrac{7p^2(2q^2 - 3q + 4q^3)}{7pq}$
23. $\dfrac{(3r^2 - 6s^2 + 9)4q}{6q}$
24. $\dfrac{6x^2(3x^3 + 9x^2 - 15x + 18)}{18x}$

50. ***Multiplying polynomials.***—The distributive law of multiplication also leads directly to the following.

To find the product of two polynomials, multiply each term of the multiplicand by each term of the multiplier and collect the products.

If the terms are all positive, this rule is represented in the figure at the right. The width of the rectangle is $a + b$ and the length is $p + q + r$. The area is $(a + b)(p + q + r)$, which the figure shows to be $ap + aq + ar + bp + bq + br$.



If $p + q + r$ is regarded as a single number S, then $(a + b)(p + q + r) = (a + b)S = aS + bS$ (the distributive law) $= a(p + q + r) + b(p + q + r) = ap + aq + ar + bp + bq + br$ (again using the distributive law).

The rule is exactly the same when some of the terms are negative.

EXERCISES

Carry out indicated multiplication.

1. $(3a - 2b)(a + 3b)$
2. $(a + b)(a - b)$
3. $(x + 2y)(2x + y)$
4. $(3x - 1)(x + 1)$
5. $(x + 5)(x - 3)$
6. $(2x - 3)(x + 1)$
7. $(a + 3)(3a - 1)$
8. $(5n + 7)(n + 2)$
9. $(3n - 4)(3n + 4)$
10. $(a + 6)(5a - 1)$
11. $(5n - 6m)(5n + 6m)$
12. $(2x + y)(3x - 2y)$

13. $(x^2 + x + 1)(x - 1)$
14. $(x^2 - x + 1)(x + 1)$
15. $(x^2 + 3x - 2)(x + 2)$
16. $(2x - 3)(x^2 - x + 2)$
17. $(2m - 3n)(m^2 - mn + n)$
18. $(3x + 2)(x^2 - 3x + 7)$
19. $(2x^2 - 1)(3x^2 + x)$
20. $(2x^2 + 3)(x^2 + 4x - 2)$
21. $(7x - 4)(x^3 - 2x + 1)$
22. $(3a - b)(2a - b + c)$
23. $(4p + q)(2p^2 - 2pq + q^2)$
24. $(x^2 + 2x + 1)(x + 1)$

25. $(x^2 - x + 1)(x^2 + x + 1)$
26. $(3x^2 - 5x + 7)(2x^3 + 8x^2 - x + 2)$
27. $(3a - 2b + 3c)(3a - 2b - 3c)$
28. $(x^4 - x^2 + 1)(x^3 + x^2 - 1)$
29. $(x^5 + x^4 + x^3 + x^2 + x + 1)(x - 1)$

51. ***Dividing polynomials.***—If we study a problem in long division in arithmetic, we shall find that the process consists in finding a series of multiples of the divisor and subtracting these from the dividend until it is exhausted, or until the last remainder is smaller than the divisor.

$$
\begin{array}{rr}
 & 723 \\
 & 37\overline{)26762} \\
700 \times 37 = & \underline{25900} \\
 & 862 \\
20 \times 37 = & \underline{740} \\
 & 122 \\
3 \times 37 = & \underline{111} \\
 & 11
\end{array}
\qquad
\begin{array}{r}
x^2 - 2x + 1 \\
x^2 - x + 4\overline{)x^4 - 3x^3 + 7x^2 - x + 2} \\
\underline{x^4 - x^3 + 4x^2} \\
- 2x^3 + 3x^2 - x \\
\underline{- 2x^3 + 2x^2 - 8x} \\
x^2 + 7x + 2 \\
\underline{x^2 - x + 4} \\
8x - 2
\end{array}
$$

Thus, in the first example above we subtract successively 700×37, 20×37, and 3×37, leaving 11 as a final remainder. We therefore conclude that $700 + 20 + 3 = 723$ is the quotient and 11 is the remainder.

A similar process is used in dividing one polynomial by another. Thus, in the second example above we subtract successively $x^2(x^2 - x + 4)$, $-2x(x^2 - x + 4)$, and $1(x^2 - x + 4)$, leaving $8x - 2$ as the last remainder.

The work of dividing will be simplified if the terms of both dividend and divisor are arranged in the order of descending exponents of some letter, as has been done in the example above.

EXERCISES IN LONG DIVISION

Divide:

1. $x^3 - y^3$ by $x - y$
2. $x^4 - 3x^2 + 2x - 1$ by $x^2 - 3x + 1$
3. $2x^3 + x^2 - 5x + 6$ by $x^2 + x + 4$
4. $3x^4 + 5x^2 + 3$ by $3x^2 + x + 3$
5. $6a^2 - 19a + 10$ by $3a - 2$
6. $8m^3 + 27n^3$ by $2m - 3n$
7. $4a^4 + 7a^3 - 5a^2 + 3a - 2$ by $2a^2 - 3$
8. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$
9. $5y^3 + 3y^2 - 4y + 2$ by $5y^2 - 1$
10. $m^3 + 3m^2 - 3m + 6$ by $m^2 - 2m + 3$
11. $2m^4 + 5m^3 - 4m^2 + 7m - 7$ by $2m^2 + 3m - 1$
12. $3a^3 + 2a - 5$ by $3a^2 - 4a - 1$
13. $5x^4 + 3x^2 + 7x - 1$ by $x^2 - 2x - 1$
14. $x^6 - 2x^4 + x^3 - 8x^2 + 8x - 2$ by $x^3 + 2x - 1$
15. $6r^4 - 3r^3 - 5r^2 + 24r - 16$ by $3r^2 + 3r - 4$.

CHAPTER 5:

SPECIAL PRODUCTS; FACTORING

In working with algebra, the products of certain factors occur often enough so that it pays to remember them, thus saving the actual work of multiplying. In arithmetic we learn the whole multiplication table for this reason.

It is also necessary to recognize certain expressions as the product of certain factors. That is, we need to recognize at sight the factors of such expressions. Can a given fraction be simplified by cancelling a common factor in the numerator and the denominator? The answer is possible only when these expressions are factored. We shall now review some of the more frequently occurring products and cases of factoring.

52. Products of the type $(a+b)(a-b)$.—By multiplying we find that the product of the sum and the difference of two numbers is the difference of their squares.

$$(a+b)(a-b) = a^2 - b^2$$

Show that the following statements are true.

$$(4x+3y)(4x-3y) = 16x^2 - 9y^2, \quad (3ab^2+7c^2d)(3ab^2-7c^2d) = 9a^2b^4 - 49c^4d^2$$
$$(a+b+c)(a+b-c) = (a+b)^2 - c^2$$
$$104 \cdot 96 = (100+4)(100-4) = 10{,}000 - 16 = 9984$$

EXERCISES

Write or read the products in the following.

1. $(2a+1)(2a-1)$
2. $(1-2a)(1+2a)$
3. $(x^2+y)(x^2-y)$
4. $(a+2b)(a-2b)$
5. $(r+a)(r-a)$
6. $(m^2-2)(m^2+2)$
7. $(3+2x)(3-2x)$
8. $(a-8b)(a+8b)$
9. $(5pq-3qr)(5pq+3qr)$
10. $(a^4-b^3)(a^4+b^3)$
11. $(ac^2-bd^2)(ac^2+bd^2)$
12. $(7m^2+3n)(7m^2-3n)$
13. $(x+y+z)(x+y-z)$
14. $(x+y+z)(x-y-z)$
15. $(x^2+x+1)(x^2-x+1)$
16. $(a+2b+3c)(-a+2b+3c)$

53. *Products of the type* $(a \pm b)^2$.—By multiplying we get the formulas at the right. That is:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

1. *The square of the sum of two numbers is the sum of their squares plus twice their product.*

2. *The square of the difference of two numbers is the sum of their squares minus twice their product.*

The right members of these equations are called the expansions of the left members. These expansions are usually written as in the above formulas, but it is convenient to think of their terms in the order in the verbal statements. These products are called trinomial squares. For the purpose of factoring, a trinomial square must be recognized as such no matter in what order its terms are given.

Example 1. $(2a+5b)^2 = (2a)^2 + 2(2a)(5b) + (5b)^2 = 4a^2 + 20ab + 25b^2$

Example 2. $(7m-2a)^2 = (7m)^2 - 2(7m)(2a) + (2a)^2 = 49m^2 - 28ma + 4a^2$

Study these examples to verify that the products as given are correct. In practice the final products are written at once, the expressions in the middle being omitted. That is, we do part of the work "in our heads."

EXERCISES

Write or read the expansions of the following.

1. $(a+1)^2$
2. $(a-1)^2$
3. $(2+x)^2$
4. $(3-x)^2$
5. $(a-2b)^2$
6. $(a+3b)^2$
7. $(4+x)^2$
8. $(x-4)^2$
9. $(4-x)^2$
10. $(2a+b)^2$
11. $(2m+3a)^2$
12. $(4+3x)^2$
13. $(3x-4)^2$
14. $(5-3b)^2$
15. $(4+5b)^2$
16. $(9-2x)^2$
17. $(2x-9)^2$
18. $(3x-7)^2$
19. $(5+8m)^2$
20. $(7a-3b)^2$
21. $(7a-3)^2$
22. $(6p+2q)^2$
23. $(8r-3s)^2$
24. $(9s+8r)^2$

25. $(2a^2+2bc)^2$
26. $(3x^2-yz)^2$
27. $(16-5abc)^2$
28. $(4a^2b+3bc)^2$
29. $(3mn-4pq)^2$
30. $(4bk^2+5m)^2$
31. $(7ab^2+4c^2d)^2$
32. $(8r^2s-3s^2t)^2$
33. $(9xy+2yz)^2$
34. $(2yz-9xy)^2$
35. $(abc+bcd)^2$
36. $(2xyz-3ax^2y)^2$
37. $(\overline{m+n}-1)^2$
38. $(\overline{m-n}-1)^2$
39. $(a^4-\overline{a^2-1})^2$
40. $(a^4+\overline{a^2-1})^2$
41. $(\overline{x^6-x^3}+1)^2$
42. $(\overline{x^6-x^3}-1)^2$

54. *Products of the type* $(x + a)(x + b)$.—In these factors, one term, x, is common to both, while the terms a and b are different. By multiplication we find the formulas below. Stated in words the first of these is:

$$(x + a)(x + b) = x^2 + (a + b)x + ab \quad (1)$$
$$(x + a)(x - b) = x^2 + (a - b)x - ab \quad (2)$$
$$(x - a)(x - b) = x^2 - (a + b)x + ab \quad (3)$$

The products of two binomials with a common term equals the square of the common term plus the sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

This holds whether the unlike terms are positive or negative. Hence formulas (2) and (3) are only special cases of (1).

Placing the factors as at the right, we regard $x \cdot x = x^2$ and $a \cdot b = ab$ as "end-products," and $a \cdot x = ax$ and $b \cdot x = bx$ as "cross-products." In the final product we then have "the sum of the end-products plus the sum of the cross-products." The sum of the cross-products can always be collected into one term.

$$\begin{array}{l} x + a \\ x + b \\ \hline x^2 + ax + bx + ab \\ x^2 + (a + b)x + ab \end{array}$$

EXERCISES

Using the above rule (formula) read or write the products in the following.

1. $(x + 3)(x + 2)$
2. $(x + 4)(x + 6)$
3. $(x - 1)(x - 3)$
4. $(x - 3)(x - 4)$
5. $(x - 4)(x + 2)$
6. $(x - 4)(x + 6)$
7. $(x + 6)(x + 1)$
8. $(x - 7)(x - 1)$
9. $(a + b)(a + c)$
10. $(a + b)(a - c)$
11. $(a - b)(a - c)$
12. $(m + 2n)(m - 3n)$
13. $(x - 4y)(x + 3y)$
14. $(2x + 1)(2x + 4)$
15. $(2x + 1)(2x - 4)$
16. $(2x - 1)(2x - 4)$
17. $(a^2 + b)(a^2 + c)$
18. $(x^2 + a)(x^2 - b)$
19. $(x^2 - a)(x^2 - b)$
20. $(mn + p)(mn + q)$
21. $(4a - 9)(4a + 12)$
22. $(3xy - 2)(3xy + 5)$
23. $(ab + c)(ab + d)$
24. $(ab - c)(ab - d)$
25. $(2mn - 1)(2mn + 5)$
26. $(7a - bc)(7a + 2bc)$
27. $(a + b + 1)(a + b - 1)$
28. $[a + (b - c)][a - (b - c)]$
29. $[(a + b) + (c + d)][(a + b) - (c + d)]$

55. ***Products of the type*** $(ax + b)(cx + d)$.—By multiplication we find the products as at the right. The product is the sum of the end-products (acx^2 and bd) plus the sum of the cross-products. The latter can always be collected into the single term $(ad + bc)x$. Note that the product of $(x + a)(x + b)$ given on page 41 can be described in exactly the words just used.

$$\begin{array}{l} ax + b \\ cx + d \\ \hline acx^2 + (ad + bc)x + bd \end{array}$$

This formula obviously holds for both negative and positive values of all the letters involved.

Example 1. $(2x+3)(4x+5) = 8x^2 + (10+12)x + 15 = 8x^2 + 22x + 15$

Example 2. $(5x-4)(2x+5) = 10x^2 + (25-8)x - 20 = 10x^2 + 17x - 20$

Example 3. $(3x-7)(2x-5) = 6x^2 + (-15-14)x + 35 = 6x^2 - 29x + 35$

In each of these the final product may be read at once.

EXERCISES

Read or write the products in the following.

1. $(x + 4)(2x + 3)$
2. $(5x + 7)(x + 2)$
3. $(2x + 1)(3x + 1)$
4. $(x - 4)(3x - 2)$
5. $(x - 7)(4x + 3)$
6. $(2x + 1)(3x - 1)$
7. $(3x - 1)(4x - 1)$
8. $(2x - 3)(4x - 7)$
9. $(2x + 3)(4x - 7)$
10. $(3x + 1)(5x - 1)$
11. $(5x - 2)(3x - 2)$
12. $(4x - 3)(5x + 2)$
13. $(3x - 2)(2x - 9)$
14. $(5x + 2)(4x - 4)$
15. $(4x + 6)(3x - 4)$
16. $(6x - 1)(2x + 5)$
17. $(2s + 3)(7s - 6)$
18. $(4m - 7)(2m + 3)$
19. $(4a + 9)(a - 12)$
20. $(6x - 7)(2x + 9)$
21. $(3x - 8)(5x - 4)$
22. $(4k - 5)(3k + 7)$
23. $(3x - 8)(5x - 8)$
24. $(3x + 8)(5x + 8)$
25. $(8x - 3)(6x + 1)$
26. $(ax + 4)(bx - 4)$
27. $(kx - 2)(lx + 8)$
28. $(mx - 3)(nx + 6)$
29. $(pq - 6)(rq + 3)$
30. $(cx - 5)(ax + 2)$
31. $(2ax - 1)(3ax + 1)$
32. $(7x + 8)(8x - 7)$
33. $(ax + b)(cx - b)$
34. $(9x - a)(7x + b)$
35. $(3ab - c)(2ab + c)$
36. $(12m - n)(am + n)$
37. $\{(a + b)x + c\}\{(a + b)x - c\}$
38. $[(2m - n)x + mn][(2m - n)x - mn]$

56. Square of a polynomial.—By multiplying we can verify the following formulas.

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$
$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$$

The general rule of which these are examples is:

The square of a polynomial equals the sum of the squares of its terms plus twice the product of each term into each term that follows it.

This rule holds whether the terms are positive or negative. Thus, $(a-b-c)^2 = a^2+b^2+c^2-2ab-2ac+2bc$.

EXERCISES

Expand the following.

1. $(a+b-c)^2$
2. $(a+b+c-d)^2$
3. $(x+y+z+4)^2$
4. $(m+n-p-q)^2$
5. $(a-b-c+d)^2$
6. $(2a-2b+c+d)^2$
7. $(a+2b-3c-d)^2$
8. $(2x+3y-2z+b)^2$
9. $(2x^2-x+2+c)^2$
10. $(ab+bc-ac+c)^2$

EXERCISES

Give the products in the following miscellaneous set.

1. $(pq-7)(pq+7)$
2. $(3x-5y)^2$
3. $(5a+3b)^2$
4. $(x-5)(x+7)$
5. $(2x-1)(3x+4)$
6. $(2a-b+3c+1)^2$
7. $(4a-7b)(4a+7b)$
8. $(4a-7b)^2$
9. $(14+3ab)^2$
10. $(3m-2n+p)^2$
11. $(5x-b)(2x+b)$
12. $(7a+8)(7a-3)$
13. $(4x-3)(2x+3)$
14. $(4x+5)(4x-1)$
15. $(5ab-7)(5ab+7)$
16. $(13-5cd)^2$
17. $(5+3a-4b+c)^2$
18. $(11+2xyz)^2$
19. $(1-8ab)(1+8ab)$
20. $(4ab+5)(4ab-3)$
21. $(5-3a^2bc)^2$
22. $(5x-3)(2x+14)$
23. $(9+5pqr)^2$
24. $(1+2x-3y+4z)^2$

25. Expand $(\overline{a+b}+c)^2$ and $(a+\overline{b+c})^2$ as squares of binomials and thus verify the expansion of $(a+b+c)^2$ given above.

26. Expand $(\overline{a+b}+\overline{c+d})^2$ as the square of a binomial and thus verify the expansion of $(a+b+c+d)^2$ given above.

27. Expand $[(a+b+c)+d]^2$ and $[a+(b+c+d)]^2$ as the squares of binomials.

57. *Factoring.*—Factoring a number or an algebraic expression consists in finding numbers or expressions whose product it is. Thus in $6 = 2 \cdot 3$ we factor 6 by finding 2 and 3, whose product it is. Similarly in $a^2 - b^2 = (a + b)(a - b)$, we factor $a^2 - b^2$ by finding $a + b$ and $a - b$, whose product it is.

Usually we do not include fractions as factors of a number. Thus, 1, 2, 3, 6 are factors of 6, but not 3/2, though $3/2 \times 4 = 6$. Obviously 1 and the number itself are factors of any number.

A number (or expression) that has no rational integral factors except itself and 1 is said to be prime. Thus 5, $x + 1$, and $x^2 + b^2$ are prime expressions.

Success in factoring algebraic expressions depends upon knowing products such as those we have just studied. When read in reverse order the formulas above are formulas for factoring.

$(a + b)x = ax + bx$	I
$(a + b)(a - b) = a^2 - b^2$	II
$(a + b)^2 = a^2 + 2ab + b^2$	III
$(a - b)^2 = a^2 - 2ab + b^2$	IV
$(x + a)(x + b) = x^2 + (a + b)x + ab$	V
$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$	VI
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$	VII

58. *Square roots of monomials.*—We may verify easily that the square root of a product is the product of the square roots of its factors.

Thus $\sqrt{4a^4b^2} = 2a^2b$, and $\sqrt{7x^6y^4z^8} = \sqrt{7}x^3y^2z^4$

The following rule may be used.

To find the square root of a monomial, find the square root of the numerical coefficient and divide by 2 the exponent of each of its literal factors.

For the present we shall not consider the case where this results in fractional exponents.

59. *Monomial factors.*—If the terms of a polynomial (including binomials) have a common factor, then this factor may be "taken out" according to formula I above. Thus

$$ax^2 + bx + dx^3 = x(ax + b + dx^2)$$

This is exactly the process described under the addition of monomials page 30, §42. To factor a polynomial, the first step is to take out any common factor that its terms may have. If the resulting factors are not prime, they should be factored.

Since an integral rational algebraic term is expressed as a product, its factors are nearly always apparent at sight. The only exception is a large number expressed in ordinary numerals, such as 297305677.

60. Factoring expressions of the type $a^2 - b^2$.—If an expression is the difference of two squares, it may be factored by using the rule:

The difference of two squares is the product of the sum and the difference of the square roots of its terms.

Thus $4 - 9a^2 = (2 + 3a)(2 - 3a)$, $16x^2 - 25a^2b^2 = (4x + 5ab)(4x - 5ab)$

EXERCISES

Factor:

1. $4a^2 - 1$
2. $1 - 4a^2$
3. $x^2 - 4a^2$
4. $4a^2 - 9b^2$
5. $m^2 - 16$
6. $9 - 4a^2b^2$
7. $m^2 - 4n^2$
8. $(a - 1)^2 - b^2$
9. $x^2 - (1 + y)^2$
10. $4m^2 - (n - 1)^2$
11. $(x + y)^2 - z^2$
12. $x^2 - (y + z)^2$
13. $(x^2 + 1)^2 - x^2$
14. $(x^2 - 1)^2 - 1$
15. $(a + b)^2 - (c + d)^2$
16. $(a + b)^2 - (c - d)^2$
17. $(a + b)^2 - (b - c)^2$
18. $(3 + x)^2 - (x - 2)^2$

61. Factoring expressions of the types $a^2 \pm 2ab + b^2$.—These are types III, IV, page 44. To factor expressions of this type it is necessary to recognize trinomial squares as such. A trinomial square has the following characteristics.

1. Two of its terms [a^2 and b^2] are squares.

2. The remaining term [$2ab$ or $-2ab$] is twice the product of the square roots of the square terms.

If this product term is positive, the given trinomial is the square of the sum of the square roots of its square terms. If this product is negative, the trinomial is the square of the difference of these square roots. We write the two factors $a + b$, $a + b$, or $(a + b)^2$; and $a - b$, $a - b$, or $(a - b)^2$. Then $a + b$ is the square root of $a^2 + 2ab + b^2$, and $a - b$ is the square root of $a^2 - 2ab + b^2$.

EXERCISES

Find the square roots of the following.

1. $x^2 + 2xy + y^2$
2. $m^2 + 4mn + 4n^2$
3. $4a^2 + 12ab + 9b^2$
4. $9b^2 - 12ab + 4a^2$
5. $16a^2b^2 + 48abc + 36c^2$
6. $36x^2 - 48xyz + 16y^2z^2$
7. $100 - 140mn + 49m^2n^2$
8. $9a^2 + 16b^2 - 24ab$
9. $25p^2q^2 + 64r^2 - 80pqr$
10. $-24mn + 144m^2 + n^2$

62. *Factoring expressions of the type* $x^2 + sx + p$.—Factoring expressions of this type depends upon V, page 44. That is, to factor $x^2 + sx + p$ we must find two numbers a and b such that $a + b = s$ and $ab = p$. If we do, then we have the required factors $x + a$ and $x + b$. In the given expression, s and p may be either positive or negative, as may be a and b.

$$\begin{aligned}(x + a)(x + b) &= x^2 + (a + b)x + ab\\ &= x^2 + sx + p\end{aligned}$$

Verify the factoring shown at the right. Also verify that in each case $a + b = s$ and $ab = p$.

$$\begin{aligned} x^2 + 3x + 2 &= (x + 2)(x + 1) \quad (1)\\ x^2 - 5x + 6 &= (x - 2)(x - 3) \quad (2)\\ x^2 + 3x - 10 &= (x + 5)(x - 2) \quad (3)\\ x^2 - 7x - 30 &= (x + 3)(x - 10) \quad (4)\end{aligned}$$

Thus in (1), $2 + 1 = 3 = s$ and $2 \cdot 1 = 2 = p$; in (4), $3 + (-10) = -7 = s$ and $3(-10) = -30 = p$.

If the factors of such an expression are not seen at once, write as shown at the right, and then try to find numbers a and b with the required properties. Such an expression may easily be a prime and hence not factorable. In this case a and b with the required properties cannot be found.

$$(x + \quad)(x + \quad)$$

As an example, try to factor $x^2 + 5x + 7$, or $x^2 - 9x + 10$.

EXERCISES

In the following, factor those that are not prime.

1. $x^2 + 8x + 15$
2. $x^2 + 2x - 15$
3. $a^2 + 3a - 15$
4. $m^2 - 10m + 15$
5. $x^2 - 5x - 14$
6. $x^2 + 12x + 27$
7. $x^2 + 13x + 40$
8. $p^2 + 13p + 40$
9. $p^2 + 10p + 21$
10. $x^2 + 5x - 9$
11. $x^2 - 3x - 40$
12. $r^2 - 6r - 12$
13. $x^2 - 10x + 21$
14. $t^2 + 12t - 64$
15. $q^2 - 4q - 21$
16. $s^2 - 11s + 30$
17. $u^2 - 12u + 30$
18. $x^2 - 9x - 14$
19. $v^2 + 3v - 40$
20. $w^2 - w - 42$
21. $c^2 + 8c - 10$
22. $x^2 + x - 27$
23. $y^2 + 6y - 27$
24. $v^2 + 5v - 14$
25. $t^2 - 16t + 40$
26. $r^2 + 9r + 14$
27. $m^2 - 13m - 42$
28. $n^2 + 4n - 21$
29. $a^2 - 7a - 8$
30. $k^2 - 12k - 64$
31. $2p - 143 + p^2$
32. $q^2 - 16q + 72$
33. $x^4 - 2x^2 - 15$
34. $m^4 + m^2 - 6$
35. $x^4 + 2x^2 - 15$
36. $m^4 - m^2 - 6$

63. *Factoring expressions of the type* $px^2 + qx + r$.—Factoring expressions of this type depends upon page 42, §55, and VI, §57. In practice it is convenient to recognize the product of $ax + b$ and $cx + d$ when these are given, but an expression of the type $px^2 + qx + r$ is usually not factorable into rational factors. The process of finding such factors, when they exist, is tedious by methods that are available at this stage. Later we shall find much more effective methods for searching for such factors; hence we shall not study this case further at this point. The following are factorable. Try to find the factors.

EXERCISES

1. $2x^2 + 5x - 3$
2. $2x^2 + 3x - 5$
3. $3x^2 + x - 2$
4. $3x^2 + 7x - 6$
5. $4x^2 + 7x - 2$
6. $2x^2 + 13x + 15$
7. $3x^2 - 16x + 21$
8. $2x^2 - 15x + 18$
9. $2x^2 + x - 15$
10. $6x^2 - 17x - 14$
11. $6m^2 - m - 15$
12. $8x^2 + 55x - 7$
13. $10p^2 + 19p - 15$
14. $14a^2 + 15a - 9$
15. $15p^2 - p - 28$
16. $18q^2 + 41q - 10$
17. $6r^2 - 11r - 10$
18. $14b^2 - 59b - 18$

64. *Factoring squares of polynomials.*—The general form of the square of a trinomial suggests that if in a polynomial of six terms three of them are squares, such as a^2, b^2, c^2, then the given polynomial may be the square of a trinomial. The terms of this trinomial must be a, b, c, and it remains to put signs between them in such a way that the products $2ab$, $2ac$, $2bc$ shall be the terms in the given polynomial. If this cannot be done, the given polynomial is not the square of a trinomial.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Example 1. $x^2 + 4y^2 + 9 + 4xy + 6x + 12y = (x + 2y + 3)^2$

Example 2. $x^2 + 4y^2 + 9 - 4xy - 6x + 12y = (x - 2y - 3)^2$

Verify the above examples.

EXERCISES

Find the square roots of the following.

1. $4a^2 + b^2 + c^2 - 4ab + 4ac - 2bc$
2. $a^2 + 4b^2 + c^2 - 4ab - 2ac + 4bc$
3. $4a^2 + b^2 + 9c^2 - 4ab + 12ac - 6bc$
4. $x^2 + 9y^2 + 4z^2 - 6xy + 4xz - 12yz$

65. Factoring expressions reducible to simpler types.—The following examples show how certain expressions may be reduced so they can be factored by the preceding methods.

Example 1. Factor $x^4 - 1$. Since x^4 is the square of x^2, this expression is the difference of two squares, and its factors are $x^2 + 1$ and $x^2 - 1$.

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$$

But $x^2 - 1 = (x + 1)(x - 1)$. Hence the factors are as shown above.

Example 2. Factor $x^4 - 5x^2 + 6$. If we replace x^2 by a, we have the expression $a^2 - 5a + 6 = (a - 2)(a - 3)$. We can easily think of the original expression in terms of x^2, and give the factors at once.

$$x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$$

Example 3. Factor $x^6 + 6x^3 - 27$. If we replace x^3 by a, we have $a^2 + 6a - 27 = (a + 9)(a - 3) = (x^3 + 9)(x^3 - 3)$.

Example 4. Factor $x^4 + x^2y^2 + y^2$. If we add and subtract x^2y^2 we have $x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - (xy)^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy)$. By this last device a few special expressions may be factored.

The expression $9x^4 + 5x^2y^2 + 25y^4$ may be factored by this method, by adding and subtracting $25x^2y^2$, giving $9x^4 + 30x^2y^2 + 25y^4 - 25x^2y^2 = (3x^2 + 5y^2)^2 - (5xy)^2$, which is the difference of two squares and is factored as below.

$$(3x^2 + 5y^2)^2 - (5xy)^2 = (3x^2 + 5y^2 + 5xy)(3x^2 + 5y^2 - 5xy)$$

This device is mainly important in that it enables us to factor $x^4 + x^2 + 1$, but not for instance $x^4 - x^2 + 1$, or $x^2 + x + 1$.

EXERCISES

Factor the following.

1. $a^4 - b^4$
2. $x^4 + 7x^2 + 12$
3. $x^6 - 9x^3 + 18$
4. $x^4 + x^2 + 1$
5. $a^8 + a^4 + 1$
6. $x^4 - 7x^2 + 1$
7. $x^8 - 1$
8. $x^4 - 6x^2 + 8$
9. $x^8 - y^8$
10. $4x^2 + y^2 + 4xy - 9$
11. $a^4 + a^2b^2 + 25b^4$
12. $16a^4 - 81b^4$
13. $ba^5 - 16ab^5$
14. $16x^4 - 81y^4$
15. $a^4 - 2a^2 - 8$
16. $x^6 + x^4 + x^2$
17. $a^8 + 2a^4 - 3$
18. $a^8b^8 + a^4b^4 + 1$
19. $a^6 - 2a^3 - 35$
20. $a^{10}b^{10} + 4a^5b^5 - 21$
21. $x^{16} - 1$
22. $x^6 - y^4$
23. $x^6 - y^6$
24. $x^4 - y^8$
25. $(x + 1)^4 - 1$
26. $a^6 - 6a^3 - 7$
27. $64a^4 - 81b^4$
28. $256k^4 - 1$
29. $a^4 + 5a^2b^2 + 6b^4$
30. $4p^2 - 8pq^2 + 4q^4$

66. Factoring expressions of the types $a^3 \pm b^3$.—By multiplying, verify the two equations at the right. These show that the sum of two cubes and also their difference can always be factored. These are special cases of factoring of $a^n \pm b^n$ studied on pages 50, 51.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 1. $(a + 1)^3 + b^3 = (a + 1 + b)[(a + 1)^2 - (a + 1)b + b^2]$
$= (a + 1 + b)[(a^2 + 2a + 1 - ab - b + b^2]$

Example 2. $(x + 2)^3 - y^3 = (x + 2 - y)[(x + 2)^2 + (x + 2)y + y^2]$
$= (x - y + 2)[x^2 + 4x + 4 + xy + 2y + y^2]$

67. Factoring by grouping.—A device for factoring shown in the following examples is sometimes convenient in reducing algebraic expressions.

Example 1.
$a^3 + a^2 - 4ab + 4b^2 - 8b^3 = a^3 - 8b^3 + (a - 2b)^2$
$= (a - 2b)(a^2 + 2ab + 4b^2) + (a - 2b)^2$
$= (a - 2b)(a^2 + 2ab + 4b^2 + a - 2b)$

Example 2. $2a^2 - 3ab + 4ac - 6bc = a(2a - 3b) + 2c(2a - 3b)$
$= (a + 2c)(2a - 3b)$

Example 3. $a^3 + b^3 + a^2 + 2ab + b^2 = (a + b)(a^2 - ab + b^2) + (a + b)^2$
$= (a + b)(a^2 - ab + b^2 + a + b)$

In each of these, success in factoring depends on grouping the terms so that each group can be factored. If this factoring shows a common factor in the result, this factor can be taken out; thus the whole expression will be factored.

EXERCISES

Factor the following.

1. $(x + y)^3 + z^3$
2. $(x - y)^3 - z^3$
3. $1 + (a + b - 1)^3$
4. $1 - (a - b + 1)^3$
5. $(x^2)^3 + (y^2)^3$
6. $x^6 - y^6$
7. $(x^3)^3 + (y^3)^6$
8. $(x^3)^2 - (y^3)^2$
9. $x^3 - y^3 + (x - y)^2$
10. $a^3 + b^3 + (a + b)^2$
11. $a^3 + b^3 + a^2 - b^2$
12. $a^3 - b^3 + a^2 - b^2$
13. $2pm - nq - np + 2mq$
14. $4xy - 4xz + 3y - 3z$
15. $abd + abk - cd - ck$
16. $2x^2 + 2xy + x + 2xy + 2y^2 + y$

68. ***The factor theorem.***—We shall indicate a polynomial in x by $p(x)$. If in $p(x)$, x is replaced by a fixed number a, we write $p(a)$ as the result. If a polynomial in x is divided by a binomial such as $x - a$, the process may be continued until a constant remainder is found. Let q be the quotient and r the remainder; then

$$\frac{p(x)}{x-a} = q + \frac{r}{x-a}, \text{ or } p(x) = q(x-a) + r.$$

This is an identity in x (see page 70, §82), since in all examples in division, multiplying the quotient by the divisor and adding the remainder gives the dividend. Hence we may substitute any value we please in $p(x) = (x - a)q + r$. Substituting $x = a$ gives $p(a) = r$, since $(a - a)q = 0$ for all values of q. Note that r does not contain x and hence remains the same no matter what number is substituted for x. Hence we see that the remainder may be found, without dividing, by substituting $x = a$ in $p(x)$.

$$\begin{aligned} p(x) &\equiv (x-a)q + r \\ p(a) &= (a-a)q + r \\ &= r \end{aligned}$$

If $r = 0$, then the division is "exact," and $p(x)$ is divisible by $x - a$. Hence we have the factor theorem:

If in a polynomial p(x) *substituting* x = a *reduces the polynomial to zero, then* p(x) *is divisible by* x − a.

This enables us to decide easily in many cases whether a given binomial is a factor of a given polynomial, as in §70.

69. ***The remainder theorem.***—From the equation $p(a) = (a - a)q + r$, it follows that the remainder r when $p(x)$ is divided by $x - a$ is equal to the value of $p(x)$ when a is substituted for x. This proposition is called the remainder theorem.

70. ***Divisibility of $x^n \pm 1$.***—Making use of the identity $p(x) \equiv (x - a)q + r$, we shall begin the study of the divisibility of $x^n \pm 1$ by means of simple examples.

Example 1. Is $x^3 - 1$ divisible by $x - 1$? Substituting $x = 1$, r is 0 and hence $x^3 - 1$ is divisible by $x - 1$.

$$\begin{aligned} &x^3 - 1 = (x-1)q + r \\ &1 - 1 = 0 + r = 0 \\ &r = 0 \end{aligned}$$

Example 2. Is $x^3 - 1$ divisible by $x + 1$? Substituting $x = -1$, $r = -2$, as shown at the right. Hence $x^3 - 1$ is not divisible by $x + 1$.

$$x^3 - 1 = (x + 1)q + r$$
$$(-1)^3 - 1 = (-1 + 1)q + r$$
$$-2 = r$$

Example 3. Is $x^4 - 1$ divisible by $x - 1$? Substituting $x = 1$ gives $r = 0$, and $x^4 - 1$ is divisible by $x - 1$.

$$x^4 - 1 = (x - 1)q + r$$
$$1 - 1 = (1 - 1)q + r$$
$$0 = r$$

Example 4. Is $x^4 - 1$ divisible by $x + 1$? Substituting $x = -1$ gives $r = 0$, and the division is possible.

$$x^4 - 1 = (x + 1)q + r$$
$$(-1)^4 - 1 = (-1 + 1)q + r$$
$$0 = r$$

We shall now study such examples in general.

Example 5. Is $x^n - 1$ divisible by $x - 1$? Substituting $x = 1$, we have $1^n - 1 = (1 - 1)q + r = 0$ for all values of n. Hence $x^n - 1$ is divisible by $x - 1$ for all values of n.

$$x^n - 1 = (x - 1)q + r$$
$$1^n - 1 = (1 - 1)q + r$$
$$0 = r$$

Example 6. Is $x^n - 1$ divisible by $x + 1$? It is evident that $(-1)^n - 1$ is zero for even values of n, and -2 for odd values of n. Hence $x^n - 1$ is divisible by $x + 1$ when n is even and not so divisible when n is odd.

$$x^n - 1 = (x + 1)q + r$$
$$(-1)^n - 1 = (-1 + 1)q + r$$

Example 7. Is $x^n + 1$ divisible by $x - 1$ or by $x + 1$? It is clear that division by $x - 1$ is never possible, and that division by $x + 1$ is possible when n is odd.

$$\left.\begin{aligned} x^n + 1 &= (x - 1)q + r \\ 1^n + 1 &= (1 - 1)q + r \end{aligned}\right\} \quad n \text{ odd or even}$$
$$2 = r$$
$$\left.\begin{aligned} x^n + 1 &= (x + 1)q + r \\ (-1)^n + 1 &= (-1 + 1)q + r \\ -1 + 1 &= (-1 + 1)q + r \end{aligned}\right\} \quad n \text{ odd}$$
$$0 = r$$

EXERCISES

1. As practice in using the method studied on pages 50, 51, test $x^2 - 1$ and $x^2 + 1$ for divisibility by $x + 1$ and by $x - 1$. How might this question be decided by reference to Example 7 above?

2. Divide $x^3 - 1$, $x^4 - 1$, $x^5 - 1$ by $x - 1$.

3. If $x^6 - 1$ is divided by $x - 1$, what is the probable quotient? Verify by multiplying or by dividing.

4. If $x^n - 1$ is divided by $x - 1$, what is the form of the quotient? Does this hold for all integral values of n?

5. If when n is an odd number $x^n + 1$ is divided by $x + 1$, what is the quotient? What is the remainder?

MISCELLANEOUS PRACTICE IN FACTORING

Factor all the expressions that are not prime.

1. $x^2 - 24x - 640$
2. $x^2 + 2x - 35$
3. $x^2 - 7x - 30$
4. $2x^2 - x - 3$
5. $6x^2 - 7x + 2$
6. $4x^2 - 5x + 7$
7. $c^2d^2 - 10cd - 96$
8. $a^4 - b^4$
9. $x^4 - 2x^2y^2 + y^4$
10. $2rw^2 - 3rw - 20r$
11. $x^4 - x$
12. $a^2 - a + 1/4$
13. $x^8 - z^8$
14. $x^6 + y^6$
15. $8 + x^2 - 2x$

16. $2(x + 2)^2 - 5(x + 2) + 3$
17. $(x^2 + 6x)^2 + 8(x^2 + 6x) + 12$
18. $(x^2 + x + 1)^2 - (x^2 + x + 1) - 2$
19. $x^4 - 14x^2y^3 + 49y^6$
20. $1/4 - 36a^2b^2$
21. $7x^2y^2 + 21xy - 14x^2y$
22. $18a^2b^2 - 12a^2b + 24ab^2$
23. $(a + w)^2 - 27$
24. $132c - 135 + 3c^2$
25. $(3x^2 - 1)^2 - (1 + 5x^2)^2$

26. $a^{12} - b^6$
27. $b^2 + 2by + y^2 - a^2$
28. $a^3 + 1/8$
29. $a^6 + 1/64$
30. $36a^4 - 16a^2 + 1$
31. $x - x^4$
32. $x^3 - x$
33. $x^3 + x$
34. $x^4 + 6x^2 + 8$
35. $x^4 - 9x^2 + 20$
36. $2a^3 + 54x^3$
37. $12x^4 + 96x$
38. $ax^2 - 2ax - 35a$
39. $ax^2 - 5ax + 6a$
40. $ab + a - 2b - 2$

41. $7x^2y^2 + 21xy + 14x^2y$
42. $2y - 3b - 4y^2 + 9b^2$
43. $(x + 2y)^3 - 8y - 4x$
44. $4a^2(2y - x) + b^2(x - 2y)$
45. $a^2 - ab - (a^2 - b^2)$
46. $a^3 - 6a^2 + 4(a - 6)$
47. $15ay - 6by + 5az - 2bz$
48. $169m^2 - 26mn + n^2$
49. $4a^2y^2 - 28ay^2 + 49y^2$
50. $x^2 + y^2 - 4xy - k^2$

51. $125v^3 - 216t^3$
52. $1/8\, x^3 - 1/27\, y^3$
53. $x^6 + y^9$
54. $1/64\, y^6 + y^9$
55. $x^4 + x^2 + 1$
56. $(a^2 + 1)^3 - b^3$
57. $b^3 - (x + 1)^3$
58. $b^3 + b$
59. $b^3 - b$
60. $(a + 1)^4 - (a + 1)$
61. $49x^2 - (y - 4)^2$
62. $x^4 - 3x^2 + 1$
63. $x^4 - 4x^2 + 1$
64. $-35x^2y + 9x^4 + 25y$
65. $a^9 + b^6$

66. $6x^2 - x - 10$
67. $x^3 - 6x^2 - 56x$
68. $(a + b)^2 + 5(a + b) - 14$
69. $(2x^2 + 3)^2 - (2 + x)^2$
70. $x^6 + x^3 + 1$
71. $x^6 + y^6 = (x^2)^3 + (y^2)^3$
72. $1/27\, x^3 - 1/8\, y^6$
73. $8a^3b^6 + 125c^6d^3$
74. $(2a + b)^4 + 8(2a + b)$
75. $(2a + b)^4 + c^4$

71. Additional products and factors.—The cubes of binomials are shown at the right. These products are special cases of the expansion of $(a+b)^n$ studied in Chapter 22. Compare the expansions in (1) and (2). Also compare the expansions of $(a+b)^2$ and $(a-b)^2$. How do the + and − signs appear in these expansions?

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad (1)$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad (2)$$

We have learned, page 51, Example 5, that $x^n - 1$ is divisible by $x - 1$ for all values of n. It is easily seen that the form of the quotient is

$$x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x + 1. \quad (1)$$

Check this statement by multiplying (1) by $x - 1$.

CHECK: $x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

We have also learned that $x^n + 1$ is divisible by $x + 1$ when n is an odd number (page 51, Example 7). In this case the quotient is of the form

$$x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \ldots + 1$$

Example 1. Factor $a^6 + 1$. Explain the solution at the right. Can $a^8 + 1$ be factored in this way?

$$a^6 + 1 = (a^2)^3 + 1$$
$$= (a^2+1)(a^4 - a^2 + 1)$$

Example 2. Factor $a^{15} + 1$ in three different ways.

$$a^{15} + 1 = (a+1)(a^{14} - a^{13} + a^{12} - \ldots + 1)$$
$$a^{15} + 1 = (a^5)^3 + 1 = (a^5+1)(a^{10} - a^5 + 1)$$
$$a^{15} + 1 = (a^3)^5 + 1 = (a^3+1)(a^{12} - a^9 + a^6 - a^3 + 1)$$

Explain the three solutions just given.

EXERCISES

1. Factor $x^9 + 1$ in different ways. Show that $x^2 - x + 1$ is a factor of this expression.
2. What factors can you find of $x^9 + 1$? of $x^9 - 1$? of $x^{10} - 1$? of $x^{10} + 1$?
3. Show that $x^3 - 1$ is a factor of $x^9 - 1$.
4. Give several even values of n for which $x^n + 1$ can be factored.
5. What factors of $x^{30} + 1$ can you find? Give two simple factors of this expression.

MISCELLANEOUS PRACTICE IN FACTORING

Factor all the expressions that are not prime.

1. $x^2 - 24x - 640$
2. $x^2 + 2x - 35$
3. $x^2 - 7x - 30$
4. $2x^2 - x - 3$
5. $6x^2 - 7x + 2$
6. $4x^2 - 5x + 7$
7. $c^2d^2 - 10cd - 96$
8. $a^4 - b^4$
9. $x^4 - 2x^2y^2 + y^4$
10. $2rw^2 - 3rw - 20r$
11. $x^4 - x$
12. $a^2 - a + 1/4$
13. $x^8 - z^8$
14. $x^6 + y^6$
15. $8 + x^2 - 2x$

16. $2(x + 2)^2 - 5(x + 2) + 3$
17. $(x^2 + 6x)^2 + 8(x^2 + 6x) + 12$
18. $(x^2 + x + 1)^2 - (x^2 + x + 1) - 2$
19. $x^4 - 14x^2y^2 + 49y^6$
20. $1/4 - 36a^2b^2$
21. $7x^2y^2 + 21xy - 14x^2y$
22. $18a^2b^3 - 12a^2b + 24ab^2$
23. $(a + w)^2 - 27$
24. $132c - 135 + 3c^2$
25. $(3x^2 - 1)^2 - (1 + 5x^2)^2$

26. $a^{12} - b^6$
27. $b^2 + 2by + y^2 - a^2$
28. $a^3 + 1/8$
29. $a^6 + 1/64$
30. $36a^4 - 16a^2 + 1$
31. $x - x^4$
32. $x^3 - x$
33. $x^3 + x$
34. $x^4 + 6x^2 + 8$
35. $x^4 - 9x^2 + 20$
36. $2a^3 + 54x^3$
37. $12x^4 + 96x$
38. $ax^2 - 2ax - 35a$
39. $ax^2 - 5ax + 6a$
40. $ab + a - 2b - 2$

41. $7x^2y^2 + 21xy + 14x^2y$
42. $2y - 3b - 4y^2 + 9b^2$
43. $(x + 2y)^3 - 8y - 4x$
44. $4a^2(2y - x) + b^2(x - 2y)$
45. $a^2 - ab - (a^2 - b^2)$
46. $a^3 - 6a^2 + 4(a - 6)$
47. $15ay - 6by + 5az - 2bz$
48. $169m^2 - 26mn + n^2$
49. $4a^2y^2 - 28ay^2 + 49y^2$
50. $x^2 + y^2 - 4xy - k^2$

51. $125v^3 - 216t^3$
52. $1/8\, x^3 - 1/27\, y^3$
53. $x^6 + y^9$
54. $1/64\, y^6 + y^9$
55. $x^4 + x^2 + 1$
56. $(a^2 + 1)^3 - b^3$
57. $b^3 - (x + 1)^3$
58. $b^3 + b$
59. $b^3 - b$
60. $(a + 1)^4 - (a + 1)$
61. $49x^2 - (y - 4)^2$
62. $x^4 - 3x^2 + 1$
63. $x^4 - 4x^2 + 1$
64. $-35x^2y + 9x^4 + 25y$
65. $a^9 + b^6$

66. $6x^2 - x - 10$
67. $x^3 - 6x^2 - 56x$
68. $(a + b)^2 + 5(a + b) - 14$
69. $(2x^2 + 3)^2 - (2 + x)^2$
70. $x^6 + x^3 + 1$
71. $x^6 + y^6 = (x^2)^3 + (y^2)^3$
72. $1/27\, x^3 - 1/8\, y^6$
73. $8a^3b^6 + 125c^6d^3$
74. $(2a + b)^4 + 8(2a + b)$
75. $(2a + b)^4 + c^4$

71. Additional products and factors.—The cubes of binomials are shown at the right. These products are special cases of the expansion of $(a+b)^n$ studied in Chapter 22. Compare the expansions in (1) and (2). Also compare the expansions of $(a+b)^2$ and $(a-b)^2$. How do the + and − signs appear in these expansions?

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad (1)$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad (2)$$

We have learned, page 51, Example 5, that $x^n - 1$ is divisible by $x - 1$ for all values of n. It is easily seen that the form of the quotient is

$$x^{n-1} + x^{n-2} + x^{n-3} + \dots + x + 1. \quad (1)$$

Check this statement by multiplying (1) by $x - 1$.

CHECK: $x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

We have also learned that $x^n + 1$ is divisible by $x + 1$ when n is an odd number (page 51, Example 7). In this case the quotient is of the form

$$x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \dots + 1$$

Example 1. Factor $a^6 + 1$. Explain the solution at the right. Can $a^8 + 1$ be factored in this way?

$$a^6 + 1 = (a^2)^3 + 1 = (a^2+1)(a^4 - a^2 + 1)$$

Example 2. Factor $a^{15} + 1$ in three different ways.

$$a^{15} + 1 = (a+1)(a^{14} - a^{13} + a^{12} - \dots + 1)$$
$$a^{15} + 1 = (a^5)^3 + 1 = (a^5+1)(a^{10} - a^5 + 1)$$
$$a^{15} + 1 = (a^3)^5 + 1 = (a^3+1)(a^{12} - a^9 + a^6 - a^3 + 1)$$

Explain the three solutions just given.

EXERCISES

1. Factor $x^9 + 1$ in different ways. Show that $x^2 - x + 1$ is a *factor* of this expression.

2. What factors can you find of $x^9 + 1$? of $x^9 - 1$? of $x^{10} - 1$? of $x^{10} + 1$?

3. Show that $x^3 - 1$ is a factor of $x^9 - 1$.

4. Give several even values of n for which $x^n + 1$ can be factored.

5. What factors of $x^{30} + 1$ can you find? Give two simple factors of this expression.

SUPPLEMENTARY PRACTICE IN FACTORING

Factor completely:

1. $a^3 - ab^2$
2. $a^2 + 3a - 28$
3. $x^3 - 1$
4. $r^2 + 5r - 84$
5. $a^3 + 8$
6. $x^2 + x + 1/4$
7. $4p^2q^2 - 25r^4s^2$
8. $x^4 + 2x^2 + 9$
9. $9 - (3x - y)^2$
10. $mn^4 - m^3n^2$
11. $1/4\, x^4 - 1/9\, y^2$
12. $1 - x^3$
13. $21ab - 6bc + 12b$
14. $x^{12} + x^6 + 1$
15. $rs^3 - 8r^2s + 16rs$
16. $9a^4 - 13a^2b^2 + 4b^4$
17. $t^2 + t - 110$
18. $a^3b - a^2b^2 + a^2b^3$
19. $a^2 - b^2 + 8b - 16$
20. $2x^3 - 20x^2 + 42x$
21. $a^2 + 12a + 36 - p^2 + 4p - 4$
22. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$
23. $16 - a^2 - 12ab - 36b^2$

24. $6x^2 - 16x^2y - 6x^2y^2$
25. $4a^2b^4 - 9c^4d^2$
26. $(a - 3)^2 - (b + 3)^2$
27. $5m^2n - 15m^3 + 10mn^2$
28. $(x + 1)^2 - (x - 1)^2$
29. $x^2 - 6x + 9 - y^2$
30. $3x^3 + 12x^2y + 12xy^2$
31. $3m^3 - 21m^2 + 18m$
32. $x^{16} + x^8 + 1$
33. $a^2 + 2a - 63$
34. $1 + x^5$
35. $8m^3 - 2mn^2$
36. $a^2 - 2a - 63$
37. $x^2 - x + 1/4$
38. $9x^2y^2 - 4m^2n^2$
39. $8a^3 + 27$
40. $r^2 - 4r - 96$
41. $4x^2 - 2x + 1/4$
42. $4x^4 - 29x^2 + 25$
43. $a^3 - 8$
44. $pq^3 + p^3q - 3p^2q^2$
45. $(a + b)^2 - a^2 + 2ab - b^2$
46. $2a^3 - 4a^2b + 2ab^2$
47. $a^2 + 12a + 36 - (p - 2)^2$
48. $x^2 - 4x + 4 - y^2 + 6y - 9$
49. $4p^4 + 24p^3 - 28p^2$
50. $p^3q + 6p^2q + 9pq$
51. $5r^3 - 50r^2 + 125r$
52. $4a^2 + 2a + 1/4$
53. $(r - s)^2 - 49 - 14s - s^2$
54. $p^2 - 2p - 80$
55. $x^6 + 1$
56. $r^2 - 20r + 99$
57. $125x^3 - 8y^6$
58. $8a^6 + 125b^3$
59. $p^2 + p - 90$
60. $27b^3 + 1$
61. $a^3 - 216$
62. $q^2 - 2q - 35$
63. $7q^5 - 42q^3 + 63q$
64. $4a^2 + 12a + 9 - 1/4$
65. $7y^3 - 42xy^2 + 63x^2y$
66. $(a + b + c)^2 - b^2 + 2bk -$

72. *Casting out 9's.*—The method for checking multiplication and division by casting out 9's, which is often used in arithmetic, depends upon certain properties of whole numbers, which we shall now study.

$$D = dq + r \quad (1)$$
$$D' = dq' \quad (2)$$
$$D - D' = d(q - q') + r \quad (3)$$

Equation (1) at the right indicates that dividing an integer D (dividend) by an integer d (divisor) gives quotient q and a remainder r, q and r being integers. We assume that the division has been carried out so that r is less than d. We then call r the excess after casting out d's. Clearly, if D is a multiple of d, then the excess is zero.

Theorem 1. If r *is the excess when casting out* d's *in a number* D *and if* D′ *is a multiple of* d, *then* r *is the excess after casting out* d's *in* D − D′.

If D' is a multiple of d, then $D' = dq'$ where q' is an integer. This theorem is then obvious from equation (3) above, which follows directly from (1) and (2).

Theorem 2. If a number N *written in the Arabic notation is divided by 9, then the remainder is equal to the remainder obtained by dividing the sum of its digits by 9.*

PROOF. Let a, b, c, d, e be the digits of the number. Then $N = 10^4a + 10^3b + 10^2c + 10d + e$. By subtracting and adding the digits of the number we have

$$N = (10^4 - 1)a + (10^3 - 1)b + (10^2 - 1)c + (10 - 1)d + (a + b + c + d + e).$$

But by Example 5, page 51, $10 - 1 = 9$ is a divisor of the first four terms of this expression. This is also seen directly since $10 - 1 = 9$, $100 - 1 = 99$, $1000 - 1 = 999$, and so on. Then by Theorem 1, the excess in N is the same as the excess in $a + b + c + d + e$, which proves the theorem.

Example. Find the excess after casting out 9's in 8356457.

SOLUTION: $8 + 3 + 5 + 6 + 4 + 5 + 7 = 38$, $3 + 8 = 11$, $1 + 1 = 2$. The sum of the digits is 38, which equals $4 \cdot 9 + 2$, and hence 2 is the required excess. But the excess in 38 can be found by adding the digits 3 and 8, and then adding 1 and 1. That is, we find the excess in 38 which by the theorem is the same as the excess in 11.

PROBLEMS

1. In a three-figure number the sum of the digits is subtracted from the number; show that the remainder is divisible by 9.

SUGGESTION: Let b, t, u be the digits. Then the number is $10^2b + 10t + u$, and $10^2b + 10t + u - (b + t + u) = (10^2 - 1)b + (10 - 1)t$ is the remainder when the sum of the digits is subtracted.

2. Prove that if the sum of the digits of a five-place number is subtracted from the number, then the remainder is divisible by 9.

SUGGESTION: Let a, b, c, d, e be the digits. Then the remainder is $(10^4 - 1)a + (10^3 - 1)b + (10^2 - 1)c + (10 - 1)d$.

3. A four-place number is divided by 9; show that the remainder is the same as the remainder when the sum of its digits is divided by 9.

SUGGESTION: If a, b, c, d are the digits of the number then $10^3a + 10^2b + 10c + d$ is the number, which may also be written $10^3a + 10^2b + 10c + d - a - b - c - d + a + b + c + d = (10^3 - 1)a + (10^2 - 1)b + (10 - 1)c + a + b + c + d$.

Clearly the first three terms of this last expression are divisible by 9.

When the sum of the digits is divided by 9 the remainder is called the excess after casting out 9's.

4. Prove the following rule for checking multiplication.

Cast out 9's in the multiplier and multiplicand, multiply the excesses and cast out 9's in the product. The excess is the same as the excess in the general product.

SUGGESTION: Denote multiplier and multiplicand by m_1 and m_2. Then $m_1 = 9q_1 + r_1$ and $m_2 = 9q_2 + r_2$ where r_1 and r_2 are the excesses in m_1 and m_2.

Hence, $m_1m_2 = (9q_1 + r_1)(9q_2 + r_2) = 81q_1q_2 + 9(q_1r_2 + q_2r_1) + r_1r_2$.

Clearly the first two terms in the right member are divisible by 9.

5. Prove the proposition of Theorem 2 on page 55 for any number whatever.

SUGGESTION: Denote the number by $10^na + 10^{n-1}b + \ldots + 10p + q$ which equals $(10^n - 1)a + (10^{n-1} - 1)b + \ldots (10 - 1)p + a + b + \ldots + p + q$.

6. Using the rule for checking multiplication given in problem 4, find a rule for checking division by casting out 9's, (a) when there is no remainder, (b) when there is a remainder.

7. Addition may be checked as indicated at the right. The steps are: *1*. Cast out 9's in the addends, 2. cast out 9's in the sum of the excesses, 3. cast out 9's in the sum 12550. These final excesses must be equal. Prove that this last statement is true if the addition is correct.

8149	$\longrightarrow$	4
3756	$\longrightarrow$	3
645	$\longrightarrow$	6
12550 $\rightarrow$ 4		13 $\rightarrow$ 4

8. Prove the statements in problems 2, 3, 4, 5 if 3 is used instead of 9.

CHAPTER 6:

FRACTIONS

The rules of operation on fractions that are used in algebra are exactly the same as in arithmetic. However, the use in algebra of letters to represent numbers brings in many complications in carrying out these operations. Moreover, in the applications of alegbra quite complicated fractions are necessary—much more complicated than those that usually occur in arithmetic. For these reasons a re-study of fractions is now necessary.

73. Common factors and reduction of fractions.—By the fundamental property of fractions both terms of a fraction may be multiplied or divided by the same factor without changing its value. Use is made of this property in reducing a fraction to its lowest terms. To do this it is necessary to find the common factors, if any, of these terms, and then divide both terms by these factors.

$$\frac{a}{b} = \frac{ma}{mb}$$

Example 1. Reduce $\frac{x^2 - a^2}{(x+a)^2}$ to its lowest terms.

SOLUTION: $\frac{x^2 - a^2}{(x+a)^2} = \frac{(x+a)(x-a)}{(x+a)(x+a)} = \frac{x-a}{x+a}$

The steps, as in any such reduction, are: 1. Factor both numerator and denominator; 2. cancel any factor that is common to both terms.

Example 2. Reduce $\frac{x^2 - 7x + 10}{x^2 + 2x - 35}$ to its lowest terms.

SOLUTION: $\frac{x^2 - 7x + 10}{x^2 + 2x - 35} = \frac{(x-2)(x-5)}{(x+7)(x-5)} = \frac{x-2}{x+7}$

Example 3. Reduce $\frac{a^3 - b^3}{a^3 + a^2b - 2ab^2}$

SOLUTION: $\frac{a^3 - b^3}{a^3 + a^2b - 2ab^2} = \frac{(a-b)(a^2+ab+b^2)}{a(a-b)(a+2b)} = \frac{a^2+ab+b^2}{a(a+2b)}$

EXERCISES

Reduce to lowest terms the fractions that are not already in lowest terms.

1. $\dfrac{2^2 \cdot 6^2 \cdot 3^4}{2^3 \cdot 6 \cdot 3^4}$

2. $\dfrac{a^4b^3c^2}{a^2b^2c^4}$

3. $\dfrac{x^2 - 2xy + y^2}{x^2 - y^2}$

4. $\dfrac{x^2 + 7x - 30}{x^2 - 7x + 12}$

5. $\dfrac{x^3 - y^3}{x^2 - y^2}$

6. $\dfrac{x^3 + y^3}{(x + y)^2}$

7. $\dfrac{64 - x^3}{16 - 8x + x^2}$

8. $\dfrac{a(x - y)^3}{(x - y)(x^2 - y^2)}$

9. $\dfrac{a^3 + 8}{a^2 - 3a - 10}$

10. $\dfrac{x^2 - 3x - 3y + xy}{(x^2 - y^2)(x - 3)}$

11. $\dfrac{3xy^4 - 3xy^2z^2}{27x^3y^2 + 27x^3yz}$

12. $\dfrac{2x^4y^6 - 20x^2y^6}{4x^3 - 42x^2 + 20x}$

13. $\dfrac{(a - b)(b - c)(c - d)}{(b - a)(c - b)(d - c)}$

14. $\dfrac{(a - b)(p + q)(x - y)}{(p^3 + q^3)(b - a)^2(x^2 - y^2)}$

15. $\dfrac{3(m - n)(b - k)(-4mn)}{12(n - m)^2(k - b)m^2n^2}$

16. $\dfrac{(r^2 - rs + s^2)(r + s)^2}{(r^3 + s^3)(r + s)^3}$

74. Common multiples.—In order to add or subtract fractions it is necessary to reduce them to a common denominator, and to do this we need to find the least common multiple of the given denominators. A product is a multiple of any of its factors. Any number is a multiple of itself and of 1. Thus 2 is a multiple of 1 and of 2; 4 is a multiple of 1, 2, and 4; and 6 is a multiple of 1, 2, 3, and 6.

If a number is a multiple of each of two or more numbers, then it is a common multiple of these numbers. The smallest common multiple is called the least or lowest common multiple (l.c.m.). The l.c.m. of algebraic expressions is the expression of lowest degree that is a multiple of each of these expressions. The numerical coefficient of the l.c.m. must be the l.c.m. of the numerical coefficients of the given expressions.

Thus, $12(x + 1)(x - 2)(x - 3)$ is the l.c.m. of $2(x + 1)(x - 2)$, $3(x + 1)(x - 3)$, and $4(x - 2)(x - 3)$.

The l.c.m. of a set of expressions contains as a factor all the factors of each expression. But this will not be true if any one of the factors in the l.c.m. is omitted.

Example 1. Find the l.c.m. of 12, 30, and 84.

SOLUTION: When the numbers are put into the factored form, the l.c.m. can be written at once. To be a multiple of 12 the number must contain the factors 2, 2, 3. To make a multiple of 30, we must annex the factor 5; and to make a multiple of 84, we must annex the factor 7. Hence $2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 420$ is the l.c.m.

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 30 &= 2 \cdot 3 \cdot 5 \\ 84 &= 2 \cdot 2 \cdot 3 \cdot 7 \\ \text{l.c.m.} &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \\ &= 420 \end{aligned}$$

If any of these factors is omitted, then the resulting product will fail to be a multiple of one or more of the given numbers.

Exactly the same idea is used in finding the l.c.m. of algebraic expressions.

Example 2. Find the l.c.m. of $a^2 - b^2$, $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$.

SOLUTION: Write each of the given expressions in the factored form.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) & (1) \\ a^2 + 2ab + b^2 &= (a + b)(a + b) & (2) \\ a^2 - 2ab + b^2 &= (a - b)(a - b) & (3) \\ \text{l.c.m.} &= (a + b)(a - b)(a + b)(a - b) \end{aligned}$$

First write $(a + b)(a - b)$, which is a multiple of (1). To make a multiple of (2) annex the factor $a + b$, and to make a multiple of (3) annex the factor $a - b$.

Show that the l.c.m. just found is $(a^2 - b^2)(a^2 - b^2)$. The only difficulty in finding the l.c.m. of a given set of expressions is to find their factors. If one or more of the given expressions cannot be factored, use such expressions as they stand as factors of the l.c.m.

EXERCISES

Find the l.c.m. Write the results in factored form.

1. 4, 7, 28
2. 15, 48, 120
3. $x^2 + 5x + 6$, $x^2 + 6x + 9$
4. $x^2 - 9x + 20$, $x^2 - 10x + 25$
5. $x^3 - y^3$, $x^2 + xy + y^2$
6. $x^3 + y^3$, $x + y$
7. $x^4 + x^2 + 1$, $x^2 + 1 - x$
8. $x - y$, $x^2 - y^2$, $x^3 - y^3$
9. $x + y$, $(x + y)^2$, $x^3 + y^3$
10. $x^3 + y^3$, $x^2 - xy + y^2$
11. $x^4 - 1$, $x^2 - 1$, $x - 1$
12. $x^4 - 1$, $x^2 + 1$, $x + 1$
13. $x^2 + 3x + 2$, $x^2 + 5x + 6$
14. $x^3 - x$, $x^2 - 5x + 4$
15. $x^2 - y^2$, $x^3 - y^3$, $x^2 - 2xy + y^2$
16. $x^2 + x - 20$, $x^2 + 2x - 15$, $x^2 + x - 12$
17. $a^3 + b^3$, $a^2 + 2ab + b^2$, $a^2 + ab - ac - bc$
18. $x^4 - 16y^4$, $x^2 + 4y^2$, $x - 2y$, $x + 2y$

75. ***Reducing fractions to a common denominator.***—The first step in reducing fractions to a common denominator is to find the l.c.m. of their denominators. This will be the lowest common denominator (l.c.d.) of the given fractions.

Example 1. Reduce $\frac{1}{x-1}, \frac{x+3}{x^2-1}, \frac{x+4}{x^2+3x+2}$ to a common denominator.

STEP 1. Factor the denominators. This gives $(x-1), (x+1)(x-1), (x+1)(x+2)$. The l.c.d. is $(x-1)(x+1)(x+2)$.

$$(1)\quad \frac{1}{x-1} = \frac{(x+1)(x+2)}{(x-1)(x+1)(x+2)}$$

$$(2)\quad \frac{x+3}{(x-1)(x+1)} = \frac{(x+3)(x+2)}{(x-1)(x+1)(x+2)}$$

$$(3)\quad \frac{x+4}{(x+1)(x+2)} = \frac{(x+4)(x-1)}{(x-1)(x+1)(x+2)}$$

STEP 2. Multiply both terms of the first fraction by $(x+1)(x+2)$, both terms of the second fraction by $x+2$, and both terms of the third fraction by $x-1$. In each case we multiply both terms of the fraction by the smallest factor that will give the required common denominator.

Example 2. Reduce to a common denominator the fractions at the right. The second column shows the denominators in the factored form. The terms of the first fraction must now be multiplied by $(x+2)(x-3)$, those of the second by $(x-2)(x-3)$, and so on.

$$(1)\quad \frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)}$$

$$(2)\quad \frac{x+1}{x^2+x-2} = \frac{x+1}{(x+2)(x-1)}$$

$$(3)\quad \frac{x-1}{x^2-x-6} = \frac{x-1}{(x+2)(x-3)}$$

$$\text{l.c.d.} = (x-1)(x-2)(x+2)(x-3)$$

In carrying out the work, the denominators should be kept in the factored form, $(x+1)(x-2)(x+2)(x-3)$.

EXERCISES

Reduce each of the following sets of fractions to a common denominator.

1. $\frac{3}{a^2-b^2}, \frac{5}{(a+b)^2}$

2. $\frac{a}{a^3+b^3}, \frac{b}{(a+b)^2}, \frac{c}{a+b}$

3. $\frac{a}{x^4-y^4}, \frac{2}{x^2+y^2}, \frac{3}{x-y}$

4. $\frac{3}{a}, \frac{2}{a-b}, \frac{1}{a+b}, \frac{4}{(a+b)^2}$

5. $\frac{4x}{x^2-6x+5}, \frac{1}{x^2+3x-4}, \frac{x+1}{(x-1)^2}$

6. $\frac{1}{x^2-y^2}, \frac{x}{x^2+2xy-3y^2}, \frac{y}{x^2+4xy+3y^2}$

76. Addition and subtraction of fractions.—In adding or subtracting fractions they must first be reduced to a common denominator, and then the numerators added or subtracted as may be required.

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$
$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

EXERCISES

1. Add the fractions in Example 1, page 60.
2. Add the fractions in Example 2, page 60.
3. In Example 1, page 60, subtract fraction (2) from the sum of fractions (1) and (3).

Add or subtract as indicated in the following.

4. $\dfrac{2}{x^2-x-6}+\dfrac{4}{x^2-4x+3}$
5. $\dfrac{1}{x^2+3x+2}-\dfrac{3}{x^2-1}$
6. $\dfrac{5}{a^2+4a-5}-\dfrac{3}{a+5}$
7. $\dfrac{2}{a^2-1}-\dfrac{1}{a-1}+\dfrac{5}{a+1}$
8. $\dfrac{x}{x-y}-\dfrac{y}{x+y}+\dfrac{xy}{x^2-y^2}$
9. $\dfrac{x}{x^2-y^2}+\dfrac{y}{x^2+2xy+y^2}$
10. $\dfrac{3}{x^2-6x}+\dfrac{2}{x^2-5x-6}$
11. $\dfrac{1}{x^2}-\dfrac{2}{xy}+\dfrac{1}{y^2}$
12. $\dfrac{a}{a+b}+\dfrac{1}{a}+\dfrac{1}{b}$
13. $\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{xy}$
14. $\dfrac{a}{a^2-b^2}+\dfrac{b}{a-b}$
15. $\dfrac{x}{y}-2+\dfrac{y}{x}$
16. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-c)(c-d)}+\dfrac{1}{(b-c)(c-d)}$

Note that the l.c.d. is $(a-b)(b-c)(c-d)$.

17. $\dfrac{1}{(x-1)(x+2)}+\dfrac{1}{(x+2)(x-3)}+\dfrac{1}{(x-1)(3-x)}$
18. From the sum of $\dfrac{4}{a^2-4a}$ and $\dfrac{9}{a^2-16}$ subtract the sum of $\dfrac{1}{a}$ and $\dfrac{1}{a+4}$.
19. What fraction must be added to $\dfrac{a^2-3a-9}{a^3-27}+\dfrac{4}{a-3}$ to make the sum $\dfrac{a-1}{a^2+3a+9}$?
20. Subtract $\dfrac{3}{5yx-2y^2-2x^2}$ from $\dfrac{2}{2x^2+3xy-2y^2}$, and $\dfrac{2}{2x^2+xy-y^2}$ from $\dfrac{4}{3yx-y^2-2x^2}$; then add the differences.

77. Multiplication and division of fractions.— The rules used in algebra are exactly the same as those used in arithmetic.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$$

State in words the substance of the formulas at the right.

Example. Multiply the fractions at the right. The first step is to factor the terms of the fractions with a view to canceling. Then cancel $x-1$ and $x+2$, and multiply. The result may be left in the factored form as shown.

$$\frac{x^2+3x-4}{x^2-x-6}\cdot\frac{x^2-4}{x^2+2x-3}$$
$$=\frac{(x-1)(x+4)}{(x+2)(x-3)}\cdot\frac{(x-2)(x+2)}{(x+3)(x-1)}$$
$$=\frac{(x+4)(x-2)}{(x-3)(x+3)}$$

EXERCISES

Perform the multiplications and divisions as indicated.

1. $\dfrac{x^2-9}{x+3}\cdot\dfrac{2x+6}{x-3}$

2. $\dfrac{x^2+x-6}{x^2+x-12}\cdot\dfrac{x^2-x-6}{x^2-x-12}$

3. $\dfrac{b^2-ab-2a^2}{a^3-9ab^2}\div\dfrac{b-2a}{3b-a}$

4. $\dfrac{x-3}{x^2-2x+4}\div\dfrac{x^2-9}{x^3+8}$

5. $\dfrac{a}{a-b}\cdot\dfrac{b}{a+b}\div\dfrac{a+b}{a(a^2-b^2)}$

6. $\dfrac{x^2-y^2}{x^2y}\cdot\dfrac{x(x-y)}{x^2+2xy+y^2}$

7. $\dfrac{3x+2}{x^2+9x+20}\div\dfrac{3x^2+5x+2}{x^2+2x-15}$

8. $\dfrac{9x^2-4y^2}{3x+2y}\div(3x-2y)$

9. $\dfrac{2+5a+3a^2}{(1+a)^2}\div\dfrac{6a^2}{1+a^2}$

10. $\dfrac{a^2-b^2+a-b}{(b-c)^2}\div\dfrac{a-b}{b-c}$

11. $\dfrac{6x^2-x-2}{4x^2+4x-3}\cdot\dfrac{2x^2-5x-12}{9x^2+6x-8}\div\dfrac{2x^2-7x-4}{6x^2+5x-4}$

12. $\dfrac{m^2-2m+4}{m-4}\cdot\dfrac{m^2+m-2}{m^2-2m+1}\div\dfrac{m^4+8m}{m^3+4m^2-5m}$

13. $\dfrac{a^3-b^3}{a(a+b)}\div\left(\dfrac{a^2b-a^2}{b^2-a^2}\div\dfrac{2a}{a^2-b^2}\right)$

14. $\dfrac{6x^2+13xy-5y^2}{2x^2-9xy+9y^2}\div\dfrac{2x^2-xy-15y^2}{6x^2-13xy+6y^2}$

15. $\left(\dfrac{3n-x}{x^2-4n^2}\cdot\dfrac{x^2-3nx+2n^2}{x-3n}\right)\div\dfrac{x-n}{(2n-x)^2}$

78. ***Reducing complex fractions.***—In practice, fractions often occur in which the terms themselves are fractions. Such fractions are said to be complex. The following are typical.

Example 1. Evaluate $\frac{a-b}{a+b}$ for $a = \frac{3}{4}$ and $b = \frac{2}{3}$.

SOLUTION: By substituting we have $\dfrac{\frac{3}{4}-\frac{2}{3}}{\frac{3}{4}+\frac{2}{3}}$.

We may now find $\frac{3}{4}-\frac{2}{3}=\frac{1}{12}$ and $\frac{3}{4}+\frac{2}{3}=\frac{17}{12}$ and then $\frac{1}{12}\div\frac{17}{12}=\frac{1}{12}\cdot\frac{12}{17}=\frac{1}{17}$.

But more simply, we may simplify the fraction by multiplying both terms by 12, $(3 \cdot 4)$.

Then $$\frac{12\cdot\frac{3}{4}-12\cdot\frac{2}{3}}{12\cdot\frac{3}{4}+12\cdot\frac{2}{3}}=\frac{9-8}{9+8}=\frac{1}{17}$$

Multiplying by 12 may be done at sight, writing $\frac{9-8}{9+8}$ at once.

A fraction of this type may be simplified by multiplying both terms by the l.c.m. of the denominators of the minor fractions.

Example 2. Evaluate $\frac{a^2-b^2}{a^2+b^2}$ for $a = \frac{3}{8}$, $b = \frac{1}{3}$.

SOLUTION: $$\frac{\frac{9}{64}-\frac{1}{9}}{\frac{9}{64}+\frac{1}{9}}=\frac{81-64}{81+64}=\frac{17}{145}$$

Solve this example by the first method used above and compare the steps in the two solutions. In some cases you may find the first method more convenient. In each case use the one that seems simpler.

Example 3. Simplify $$\frac{\frac{a^2-1}{a^2+1}-\frac{a^2+1}{a^2-1}}{\frac{a^2-1}{a^2+1}+\frac{a^2+1}{a^2-1}}$$

SOLUTION: Multiply both terms by $(a^2+1)(a^2-1)$. This gives

$$\frac{(a^2-1)^2-(a^2+1)^2}{(a^2-1)^2+(a^2+1)^2}=\frac{a^2-2a+1-a^2-2a-1}{a^2-2a+1+a^2+2a+1}=\frac{-4a}{2a^2+2}=-\frac{2a}{a^2+1}$$

It may happen that the terms of a fraction are mixed numbers as in $\dfrac{a - \frac{1}{a}}{a + \frac{1}{a}}$. In this case we may reduce the terms to $\dfrac{a^2 - 1}{a}$ and $\dfrac{a^2 + 1}{a}$ and then divide. Or we may multiply both terms of the given fractions by a, obtaining the final result at once.

$$\boxed{\dfrac{a - \frac{1}{a}}{a + \frac{1}{a}} = \dfrac{a^2 - 1}{a^2 + 1}}$$

EXERCISES

Simplify:

1. $\dfrac{\dfrac{8n^3 + 1}{2n^2 + 4n}}{\dfrac{4n^2 - 2n + 1}{n^2 + 4n + 4}}$

2. $\dfrac{\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2}}{\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2}}$

3. $\dfrac{\dfrac{x + b + 2}{x + b} - \dfrac{x + 2}{x}}{b}$

4. $\dfrac{1 - \dfrac{x-1}{x+1}}{1 + \dfrac{x+1}{x-1}}$

5. $\dfrac{\dfrac{2x}{3y} + \dfrac{3y}{2x}}{6xy - \dfrac{5}{12xy}}$

6. $\dfrac{a^3 - \dfrac{1}{a^3}}{a^2 + 1 + \dfrac{1}{a^2}}$

7. $\dfrac{2 - \dfrac{x}{x+y}}{1 - \dfrac{2x}{x-y}}$

8. $\dfrac{\left(x + \dfrac{1}{n}\right)^2 - x^2}{\dfrac{1}{n}}$

9. $\dfrac{\dfrac{3w^2 - 5wz + 2z^2}{16w^2 - 8wz + z^2}}{\dfrac{27w^3 - 8z^3}{4w^3 - w^2z}}$

10. $\dfrac{1 + \dfrac{1}{a} - \dfrac{1}{a^2}}{\dfrac{1}{a^2} + \dfrac{2}{a^3} - \dfrac{3}{a^4}}$

11. $\dfrac{1 - \dfrac{1}{x+2}}{2x - 2 - \dfrac{12}{x-2}}$

12. $\dfrac{\dfrac{x^2+2}{x^2-1} + \dfrac{1}{1-x}}{1 + \dfrac{x^2}{x^4+1}}$

13. $\dfrac{2 - \dfrac{1}{2-a} + a}{a + \dfrac{1}{a+2} - 2}$

14. $\dfrac{\dfrac{1}{1+x} + \dfrac{1-x}{x}}{\dfrac{x}{1+x} - \dfrac{1-x}{x}}$

15. $\dfrac{1 + \dfrac{2}{x-1}}{\dfrac{x^2+x}{x^2+x-2}}$

16. $\dfrac{1 + \dfrac{1}{x+2}}{2x - 2 + \dfrac{12}{x-2}}$

79. ***A special type of fractions.***—Fractions of the type shown in the following example occur in connection with the subject of continued fractions.

Example. Simplify $1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{a}{1-a}}}$

In this case a little different method is used, as follows.

$$1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{a}{1-a}}}=1-\cfrac{1}{2-\cfrac{1}{\cfrac{3-4a}{1-a}}}=1-\cfrac{1}{2-\cfrac{1-a}{3-4a}}$$

$$=1-\cfrac{1}{\cfrac{6-8a-1+a}{3-4a}}=1-\frac{3-4a}{5-7a}=\frac{5-7a-3+4a}{5-7a}=\frac{2-3a}{5-7a}=\frac{3a-2}{7a-5}$$

Check all steps in the above solution.

EXERCISES

Simplify:

1. $1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{x}}}$

2. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}$

3. $\dfrac{\dfrac{a}{a-b}-\dfrac{b}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$

4. $\cfrac{1}{x+\cfrac{1}{1+\cfrac{x+1}{3-x}}}$

5. $\cfrac{a}{1+\cfrac{1}{1+\cfrac{1}{a}}}+\cfrac{a}{1-\cfrac{1}{1-\cfrac{1}{a}}}-\cfrac{a}{1+\cfrac{1}{1-\cfrac{1}{a}}}$

6. $\left(x-\cfrac{1}{1+\cfrac{1}{x-1}}\right)\left(x-\cfrac{1}{1-\cfrac{1}{x+1}}\right)\div\left(x+\cfrac{1}{1+\cfrac{1}{x+1}}\right)$

7. $1+\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1-a}}}$

8. $1+\cfrac{2}{2+\cfrac{1}{3-\cfrac{3}{4-\cfrac{1}{a}}}}$

SPECIAL PRACTICE ON FRACTIONS

Reduce all fractions to lowest terms.

1. $\frac{18}{45}$ 2. $\frac{5x^2}{15ax}$ 3. $\frac{3x^2+7x}{5x^2-9x}$ 4. $\frac{x^2-y^2}{(x+y)^2}$ 5. $\frac{(x-y)^2}{x^2+xy-2y^2}$

6. $\frac{3}{7}\cdot\frac{21}{18}$ 7. $\frac{x}{2}\cdot\frac{4a+2b}{(3a-2b)x}$ 8. $\frac{3a^2b}{4c^3d}\cdot\frac{8c^2d^4}{6ab^2}$

9. $\frac{4a^2-9b^2}{4a^2-8ab+3b^2}$ 10. $\frac{a}{a+b}\cdot\frac{a^2-b^2}{b(a-b)}$ 11. $\frac{a^3-b^3}{a-b}\cdot\frac{a}{a^2+ab+b^2}$

In each set change each fraction so the expression at the right end of the line will be the denominator.

	Denominators
12. $\frac{1}{4a^3b^3xy^5}, \frac{5}{6ab^4x^2y^2}, \frac{2}{3b^4x^3}, \frac{7}{6a^3bxy^3}$	$12a^3b^4x^3y^5$
13. $\frac{2(x+2)}{(x-1)(x-2)}, \frac{3(x-1)}{4(x-2)(x-3)}, \frac{5(x-3)}{8(x-1)(x-3)}$	$8(x-1)(x-2)(x-3)$
14. $\frac{1}{(a-b)(a-b)}, \frac{2}{(a^2-b^2)(a+b)}, \frac{3}{a^2-b^2}$	$(a-b)^2(a+b)^2$

Find the l.c.m. in each of the following.

15. 2, 4, 7, 8, 10 16. $a-b,\ a+b,\ (a+b)^2,\ (a-b)^2,\ a^2-b^2$
17. $a^2+a+1,\ a^2-a+1,\ a^4+a^2+1$
18. $(a+b+c)^2,\ (a+b)^2-c^2,\ a+b-c$
19. $mn^4-m^3n^2,\ mn,\ m+n,\ m-n,\ m^2-n^2$
20. $4p,\ p^2,\ p^2+6p-1,\ p^3+6p^2-p,\ 4p^4+24p^3-28p^2$

Reduce to common denominators.

21. $\frac{2}{(x^2-y^2)}, \frac{3}{(x+y)^2}, \frac{4}{(x-y)^2}, \frac{1}{(x^2-y^2)(x+y)}$

22. $\frac{a}{a^2-3a+2}, \frac{a-1}{a^2-5a+6}, \frac{a+1}{a^2-4a+3}$

23. $\frac{a}{a-1}, \frac{1}{a^3-1}, \frac{2}{a^2+a+1}$

24. $\frac{1}{x-1}, \frac{2}{x+1}, \frac{x}{x^2+1}, \frac{x^2}{x^4-1}, \frac{x}{x^2-1}$

In each of the following remove as many negative signs as possible without changing the sign of the fraction.

25. $\frac{2(-x)y(-z^2)}{-yx(-y)(-z)}$ 26. $\frac{(-x)^2(-y)(-z)}{-x\cdot y(-z)^3}$ 27. $\frac{a(-b)(-c)(-d)}{-p(-q)(-r)st}$

SUPPLEMENTARY PRACTICE ON FRACTIONS

Remove all negative signs from the terms of these fractions, prefixing each fraction with a + or a − sign as may be necessary.

1. $\frac{(-m)^2(-n)(-p)^3}{a(-b)(-c)^2}$ 2. $\frac{(-r)(-s)^2(-t)^4}{(-p)(-q)(-u)}$ 3. $\frac{(-r)(-s)^3(-t)}{(-p)^2(-q)(-u)^3}$

In the following, change the order of the letters in each factor so that they will be in alphabetic order.

Thus $\frac{(a-b)(c-b)(d-a)}{(a-c)(b-c)(b-a)} = -\frac{(a-b)(b-c)(a-d)}{(a-c)(b-c)(a-b)}$

Note that the order has been changed in three factors and that hence the sign of three factors has been changed. Therefore the sign of the fraction is changed.

4. $\frac{(p-q)(r-q)(s-r)}{(y-x)(z-x)(z-y)}$ 5. $-\frac{(a-b)(a-c)(b-c)}{(z-x)(z-y)(y-z)}$

Perform the following indicated operations and reduce results to the simplest form.

6. $1 \div \left(\frac{1}{x} - \frac{1}{y}\right)$
7. $\left(\frac{a}{b} - \frac{b}{a}\right) \div \left(\frac{1}{a} + \frac{1}{b}\right)$
8. $\left(\frac{m}{n} + \frac{n}{m}\right) \div \left(\frac{m}{n} - \frac{n}{m}\right)$
9. $\left(\frac{m}{n} - \frac{n}{m}\right) \div \left(\frac{m^2}{n} - \frac{n^2}{m}\right)$
10. $\left(a + 1 + \frac{1}{a-1}\right)\left(\frac{1}{a} - \frac{1}{a^2}\right)$
11. $\left(x - \frac{1}{x}\right)\left(2 - \frac{1}{x-1}\right)$
12. $\left(a + 1 + \frac{1}{a}\right)\left(a - 1 + \frac{1}{a}\right)$
13. $\left(\frac{m}{n} + 2 + \frac{n}{m}\right)\left(\frac{m}{m^2 - n^2}\right)$
14. $\left(\frac{1}{x^2} - \frac{1}{y^2}\right)\left(\frac{x}{y} + \frac{y}{x}\right)$
15. $\left(\frac{p^2}{q} + 2p + q\right)\left(\frac{q^2}{p+q}\right)$
16. $\left(1 + \frac{x}{x-y}\right)\left(3 + \frac{1}{x+1}\right)$
17. $\left(2a - \frac{1}{2a}\right) \div \left(\frac{2a-1}{a}\right)$
18. $\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) \div \frac{a^2+b^2}{ab}$
19. $\left(k - 4 - \frac{4}{k-4}\right)\left(\frac{k^2-16}{k^2}\right)$
20. $\left(r + s + \frac{s^2}{r-s}\right) \div \left(\frac{r}{r^2 - s^2}\right)$
21. $\left(\frac{1}{y^2} - \frac{1}{x^2}\right)\left(1 - \frac{1}{xy+1}\right)$
22. $(a - 3 + b) \div \left(\frac{1}{a} + 3 + \frac{1}{b}\right)$
23. $\left(p + 2 + \frac{5}{p-2}\right)\left(\frac{p-2}{p-1}\right)$
24. $\left(a - b + \frac{b^2}{a+b}\right) \div \frac{a}{a^2 - b^2}$
25. $\left(1 + \frac{x}{x-y}\right) \div \left(1 - \frac{x}{x+y}\right)$
26. $\left(a - 1 + \frac{1}{a+1}\right) \div \left(\frac{1}{a} + \frac{1}{a^2}\right)$
27. $\left(\frac{x^2}{y} - 2x + y\right) \div \frac{(x-y)^2}{y^2}$

PRACTICE IN REDUCTION OF COMPLEX FRACTIONS

Reduce:

1. $\dfrac{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}$

2. $\dfrac{\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}}{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}$

3. $\dfrac{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}{\dfrac{1}{a^2-1}}$

4. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{1}{a-b}+\dfrac{1}{a+b}}$

5. $\dfrac{\dfrac{p-q}{p+q}+\dfrac{p+q}{p-q}}{\dfrac{p}{p+q}+\dfrac{q}{p-q}}$

6. $\dfrac{\dfrac{x-3}{x-2}+\dfrac{x+1}{x+2}}{\dfrac{x^2-3x+6}{x^2-4}}$

7. $\dfrac{\dfrac{a}{b}-2+\dfrac{b}{a}}{\dfrac{a}{b}+2-\dfrac{b}{a}}$

8. $\dfrac{\dfrac{x+1}{x-4}+\dfrac{x-5}{x+5}}{\dfrac{x-1}{x+5}-\dfrac{x-2}{x+3}}$

9. $\dfrac{\dfrac{m}{n}-1+\dfrac{n}{m}}{\dfrac{m^3+n^3}{m^2n+mn^2}}$

10. $\dfrac{\dfrac{x+d+1}{x+d-1}-\dfrac{x+1}{x-1}}{d}$

11. $\dfrac{\dfrac{x+d-1}{x+d+1}-\dfrac{x-1}{x+1}}{d}$

12. $\dfrac{\dfrac{(x+d)^2+1}{(x+d)^2-1}-\dfrac{x^2+1}{x^2-1}}{d}$

13. $\dfrac{\dfrac{a^4-b^4}{a^2-2ab+b^2}}{\dfrac{a^2+ab}{a-b}}$

14. $\dfrac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}}$

15. $\dfrac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}$

16. $\dfrac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}$

17. $3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}$

CHAPTER 7:

VARIABLES; EQUATIONS

The concept of a "variable" really has its beginnings in early arithmetic. In fact, the only new element that we meet at the outset in connection with variables is the peculiar notation of which algebra makes systematic use. The concept, as well as the form, of an equation is also present in arithmetic. We shall now proceed to study variables and equations more systematically.

80. *Variables, range of a variable.*—In stating the commutative law of addends, the letters a and b represent any numbers whatever. For this reason these symbols are called variables. We shall use the following definition.

$$a + b = b + a$$

A symbol that may represent any one of a certain set of numbers is called a variable.

The set of numbers that are values which may be taken by a variable is called the range of the variable.

Thus, in $a + b = b + a$, the range of each variable is the whole system of numbers, while in a/b the range of b is restricted so as not to include zero. In the series

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}},$$

the variable n can take only positive integral values, since a series can have only an integral number of terms. That is, the range of the variable n in this case is the set of positive integers.

In the statement, "The sum of any two numbers is the same in whichever order they are taken," the words "any two numbers" are variables exactly as are a and b in $a + b = b + a$, and the range is exactly the same in the two statements.

The above definition of a variable must be used with a certain caution. Thus in the equation $ax + by = c$, x and y are to be regarded as variables and a, b, c as constants. This means that a, b, and c must be given fixed values, while x and y are then allowed to "vary" subject to the condition imposed by the equation.

81. *Equations.*—When two number expressions are connected by the equality sign, the result is called an equation. We shall note here three general kinds of equations, of which $a + b = b + a$, $x + 4 = 10$, and $A = bh$ are examples. The first is a general rule that holds for all values of the letters involved. Such an equation is called an identity.

$$a + b \equiv b + a$$
$$x + 4 = 10$$
$$A = bh$$

In the second equation the two members are equal only on condition that $x = 6$. This is called a conditional equation, or simply an equation.

The last of our three equations, the rule for finding the area of a rectangle, is a rule or formula. Any values, positive in this case, may be given to b (base) and h (altitude), and then the formula says that the area is the product of these.

The part of an equation before the equality sign is called the first member of the equation, and the part after the equality sign is called the second member.

82. *Permissible substitutions; identities.*—We have seen (page 16, §20) that division by zero is permanently excluded. Hence we are not permitted to substitute for a letter or letters in an algebraic expression any values that make a divisor equal to zero. Any substitution in an algebraic expression that does not make a divisor zero is said to be permissible.

In equation (1) at the right, any values of a and b are permissible substitutions, while in (2) neither $a = 0$ nor $b = 0$ is a permissible substitution. Both of these equations hold for all permissible substitutions for a and b.

$$a + b \equiv b + a \quad (1)$$
$$\frac{1}{a} + \frac{1}{b} \equiv \frac{a + b}{ab} \quad (2)$$

An equation is an identity in certain letters (possibly only one letter) if its members are equal for all permissible substitutions for these letters.

Statements like $3 = 3$ or $17 = 17$ are also called identities.

The symbol $\equiv$ is often used instead of $=$ when we wish to call special attention to the fact that an equation is an identity. Equations V–IX, XI, XII in §22 are all identities, as are the equations below showing the fundamental operations on fractions. The equations in Chapter 5 showing products and factors are all identities.

83. *Equivalent equations.*—Two equations containing the same variables (or unknowns) are said to be equivalent in these variables if they are satisfied by the same values of these variables (or unknowns). Thus equations (1) and (2) at the right are equivalent in a and b for any values of k.

$$a + b = b + a \qquad (1)$$
$$a + b + k = b + a + k \qquad (2)$$

Again, in the second set of equations at the right, any pair of the five equations, (1), (2), . . . , (5), are equivalent since each of these equations is satisfied by $x = \frac{1}{2}$ and by no other value of x. Under these conditions we say that the set of equations (1), . . . , (5) is an equivalent set. If in a set of equivalent equations it is known that a certain value, or values, of one or more variables will satisfy it, then we know that the same values will satisfy any other equation of the set.

$$7x - 8 + 2x = 3x - 5 \qquad (1)$$
$$9x - 8 = 3x - 5 \qquad (2)$$
$$6x = 8 - 5 \qquad (3)$$
$$6x = 3 \qquad (4)$$
$$x = 1/2 \qquad (5)$$

84. *Transformation of equations.*—To transform an equation into another that is equivalent to it we may:

1. Add the same number (number expression) to both members.
2. Subtract the same number from both members.
3. Multiply both members by the same number, except by zero.
4. Divide both members by the same number, except by zero.

Let A and B be any two algebraic expressions such that $A = B$ for certain values of the variables (or unknowns) involved, and let a be any algebraic expression, possibly an ordinary number. Then it follows from the uniqueness of the four fundamental operations that the set of five equations, (1), . . . , (5), at the right is an equivalent set. However $a = 0$ must be excluded in (4) and (5). Obviously, we may start with any one of these five equations and derive any other equation of the set by using one or more of these steps.

$$A = B \qquad (1)$$
$$A + a = B + a \qquad (2)$$
$$A - a = B - a \qquad (3)$$
$$aA = aB \qquad (4)$$
$$\frac{A}{a} = \frac{B}{a} \qquad (5)$$

Changing an equation by using these steps is called transforming the equation.

85. ***The arithmetic of fractions.***—We shall now use equations in proving certain well-known properties of fractions.

$$\frac{a}{b} = \frac{ma}{mb}$$

1. Multiplying (or dividing) both terms of a fraction by the same number does not change its value. This fundamental rule is stated in the equation at the right.

$$\frac{ma}{mb} = \frac{a}{b}$$

Let $a/b = q$. Then $a = bq$ and $ma = mbq$ (multiplying both members of the equation by m). Hence $ma/mb = q = a/b$ (dividing both members by mb).

2. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$. Let $\frac{a}{c} + \frac{b}{c} = s$. Then $c\left(\frac{a}{c} + \frac{b}{c}\right) = c \cdot \frac{a}{c} + c \cdot \frac{b}{c} = a + b = cs$, and $\frac{a+b}{c} = s$.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Let $\frac{a}{b} \cdot \frac{c}{d} = p$. Then $bd\left(\frac{a}{b} \cdot \frac{c}{d}\right) = bdp$, or $b \cdot \frac{a}{b} \cdot d \cdot \frac{c}{d} = ac = bdp$, and $p = ac/bd$.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

4. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$. Let $\frac{a}{b} \div \frac{c}{d} = q$. Then $\frac{a}{b} = \frac{c}{d} \cdot q$. Hence $ad = bcq$, and $q = \frac{ad}{bc}$.

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

Give the authority for each of these steps.

86. ***Solving equations.***—In a conditional equation, a letter whose value is to be found is called the unknown. The problem then is to find a value of this unknown such that substituting it in the original equation will reduce both members to the same number, or will reduce the equation to an identity. Such a value of the unknown is called a solution or root of the equation.

We shall study the solution shown at the right. The successive steps are taken for the purpose of obtaining an equation whose first member will be x and whose second member will not contain x.

$$x - 3 + 2(x + 3) = x + 9 \quad (1)$$
$$x - 3 + 2x + 6 = x + 9 \quad (2)$$
$$3x - x = 9 - 6 + 3 \quad (3)$$
$$2x = 6 \quad (4)$$
$$x = 3 \quad (5)$$

From §83 we know that the equations in the set (1), . . . , (5) are equivalent. But $x = 3$ (and no other number) satisfies (5). Hence we know that $x = 3$ (and no other number) satisfies (1). Hence $x = 3$ is the solution we are seeking.

To make certain that no error has been made in the solution we substitute $x = 3$ in both members. Since this reduces the two members to the same number, 12, we know that the solution is correct.

$$3 - 3 + 2(3 + 3) = 3 - 3 + 12 = 12$$
$$\text{Also} \quad 3 + 9 = 12$$

87. *Transposing.*—The four fundamental operations given on page 71, §84, are the main steps used in solving equations, but there is one modification for immediate use that we shall now notice. (For one additional operation see pages 142, 143.) In the solution shown at the right, equation (2) is obtained from (1) by adding 4 to each member and subtracting $3x$ from each member. In practice these operations are described by saying that we transpose these terms. In each case, the term is made to appear in the opposite member of the equation with its sign changed. That is, we "transpose" -4 from the first to the second member, writing $+4$. This amounts simply to adding 4 to both members, and noting that $-4 + 4 = 0$. The term $3x$ is "transposed" from the second to the first member, being written $-3x$. This amounts to subtracting $3x$ from both members.

$$7x - 4 = 3x + 16 \quad (1)$$
$$7x - 3x = 16 + 4 \quad (2)$$
$$4x = 20 \quad (3)$$
$$x = 5 \quad (4)$$

The above illustrates the following rule.

In an equation any term may be transposed by changing its sign.

This rule is merely a summary of the two rules for adding the same number to both members and subtracting the same number from both members.

EXERCISES

Solve each of the following equations and give the authority for each step. Check the solutions by substituting in the original equations.

1. $7x - 12 + 3(x - 3) = 5x + 4$
2. $18 - 3x + 8(x + 1) = 2x + 35$
3. $3 + 6(x - 3) + x = 5x - 5$
4. $9 - 2x + 3(x - 4) = 3x - 23$
5. $12x - 8(x + 3) = 5x - 31$
6. $21 + 3(x - 10) + x = 3(x + 3)$
7. $4x + 5(8 - x) = 5(4 - x)$
8. $14 - 9(x - 4) = 12x - 76$

88. Multiplying by zero in solving equations.—We shall note how in solving an equation both members may inadvertently be multiplied by zero. Starting with equation (1) at the right and multiplying by $x - 2$, we obtain equations (2), (3), (4), which are satisfied by $x = 2$, $x = 3$, while (1) is satisfied only by $x = 3$. Hence the extra root $x = 2$ is introduced by multiplying by $x - 2$, which is 0 for $x = 2$.

$$x + 4 = 7 \quad (1)$$
$$(x + 4)(x - 2) = 7(x - 2) \quad (2)$$
$$x^2 + 2x - 8 = 7x - 14 \quad (3)$$
$$x^2 - 5x + 6 = 0 \quad (4)$$
$$x = 2, \qquad x = 3$$

In general, if both members of an equation are multiplied by an expression M containing x, then any value of x that reduces M to zero will satisfy the resulting equation. Hence such multiplication introduces as a new (and incorrect) solution any value of x that reduces M to zero. The reason is that for such values of x we have really multiplied by zero. In the case of equations containing fractions there is an obvious modification of this statement, which we shall note on page 78, §91.

89. Dividing by zero in solving equations.—Let us start with equation (2) above, now marked (1). Dividing both members by $x - 2$, we have equation (2), which is satisfied by $x = 3$ but not by $x = 2$, while (1) is satisfied by both of these. That is, dividing both members by $x - 2$ loses the solution $x = 2$.

$$(x + 4)(x - 2) = 7(x - 2) \quad (1)$$
$$x + 4 = 7 \quad (2)$$
$$x = 3 \quad (3)$$

If the two members of an equation in x are divided by an expression D containing x, D being a factor of both members, then any value of x that reduces D to zero is lost as a root.

Example. Solve $2(3x - 2) + 3(x + 4) + 6x - 10 = 7(x - 3) + 51$

SOLUTION: The steps in the solution are shown at the right. Describe each step and give the rule that is used.

$$2(3x - 2) + 3(x + 4) + 6x - 10 = 7(x - 3) + 51 \quad (1)$$
$$6x - 4 + 3x + 12 + 6x - 10 = 7x - 21 + 51 \quad (2)$$
$$15x - 2 = 7x + 30 \quad (3)$$
$$8x = 32 \quad (4)$$
$$x = 4 \quad (5)$$

In the solution just given, the members of the equations have not been multiplied or divided by any expression that can have the value zero. Hence the five equations shown in the solution are equivalent, and $x = 4$ must satisfy equation (1) provided the work has been done correctly. The result may be checked either by substituting in the original equation or by going over the work carefully. In some cases in which the solution is a complicated expression, substitution is tedious and going over the work again is much easier.

EXERCISES

Solve the following equations. Check the first ten by substitution, and the others by repeating your work.

1. $3x - 8 = 12 - x$
2. $15 - 6x = 21 - 8x$
3. $7x + 12 = 12x - 13$
4. $4(2x + 1) = 3(7 - x) + 5$
5. $5x - 12 = 3(4x - 5) - 11$
6. $5x + 3(2 - x) = 3(x + 1) + 4$
7. $2.5x - 3(4 - x) = -1$
8. $\frac{1}{2}x + \frac{3}{4}x + 2\frac{1}{2}x = 30$
9. $1\frac{1}{3}x - 7 = 5\frac{3}{4}x - 60$
10. $x + 7 + 3x + 6x = 47$

11. $3(5x - 4) + 7 + 4x = 107 + 3x$
12. $2.4x + 1.3x + 5 = 2.1x + 24$
13. $3(x + 2) - 2(x - 1) + 15 = 4(x - 3) + 16$
14. $5(x + 3) + 25 = 3(x + 1) + 4(x + 2) - 3x$
15. $45 + 2(x - 3) + 4x - 14 = 3(x + 5) + 13$
16. $4.7x + 4(x - 3) - 2.5x = 6.1x - 5$
17. $4(x + 5) + 2.7 = 19 + 2(x + 6) - 4x + 7$
18. $7(4x - 3) + 2(4 - x) + 5 = 5(3x + 1)$
19. $4 - 3(5 - 2x) + 6(3x - 2) = 2(1 - x) + 8$

In the following, terms containing x^2 are made to disappear by subtracting equal terms from both members.

20. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$
21. $(3x - 1)(18 - x) = (x + 6)(16 - 3x)$
22. $(4x + 4)(x - 3) = (4x + 1)(x + 7) - 13x + 221$
23. $(x - 1)(13 - 6x) = (6x - 3)(8 - x) - 21$
24. $(7x - 13)(6 - x) = (x + 4)(3 - 7x) + 70$
25. $(x - 1)(3x - 1) = (x + 1)^2 + 2x^2 - 18$
26. $(6 - x)^2 + (x - 3)(2x - 5) = (3x + 1)(x - 3) + 84$
27. $x^2 = (x - 3)(x + 6) - 12$
28. $x(x + 3) + (x + 1)(x + 2) = 2x(x + 5) + 2$
29. $(x + 6)^2 + 3(x - 1)^2 = 25x + 9 - (4 - x)(4x + 6)$

90. Solving equations containing fractions.—The first step in solving an equation containing fractions is to clear the equation of fractions. This reduces the equation to the case we have already studied. The idea in its simplest form is illustrated in the examples at the right.

I

$$x/4 = 6$$
$$x = 24$$

In solving $x/4 = 6$, we multiply both members by 4.

In solving $\frac{x}{2} + \frac{x}{3} + \frac{x}{6} = 1$, we may proceed in one of two ways.

II

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{6} = 1 \quad (1)$$
$$6\frac{x}{2} + 6\frac{x}{3} + 6\frac{x}{6} = 6 \quad (2)$$
$$3x + 2x + x = 6 \quad (3)$$
$$6x = 6 \quad (4)$$
$$x = 1 \quad (5)$$

(1) We may write the equation thus: $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{6}x = 1$, $\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)x = 1$. Adding the fractions gives $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, and the equation reduces to $x = 1$. In more complicated cases this method becomes tedious, or even impossible.

(2) We may multiply both members of the given equation by a number that will enable us to cancel the denominators 2, 3, and 6. Clearly the smallest multiplier that can be used is 6. The process is shown in II above.

III

$$\frac{x-3}{3} + \frac{x+4}{4} = 3 \quad (1)$$
$$12 \cdot \frac{x-3}{3} + 12 \cdot \frac{x+4}{4} = 36 \quad (2)$$
$$4(x-3) + 3(x+4) = 36 \quad (3)$$
$$4x - 12 + 3x + 12 = 36 \quad (4)$$
$$7x = 36 \quad (5)$$
$$x = 36/7$$

In III at the right both members are multiplied by 12, since this is the smallest number that will enable us to cancel the denominators 3 and 4.

In practice we cancel "mentally" as we multiply. Thus in II, we write equation (3) directly from (1), as we do also in III. This is the process that is always used in practice.

The process just described is called clearing an equation of fractions. The rule for doing this is:

Multiply both members of the equation by the least common multiple of the denominators of its fractions.

Much trouble is saved by multiplying by the *lowest* common denominator and not by some higher common denominator.

Study the solutions below. As in the examples on page 76, so in I below, equation (2) may be omitted, (3) being obtained directly from (1) by thinking the canceling in the fractions as we multiply both members by $(x-4)(x+2)(x+1)$. In II, by factoring the denominators, we see that $(x-1)(x-2)(x+1)(x+2)$ is the l.c.m. of the denominators.

I

$$\frac{2}{x-4}+\frac{3}{x+2}=\frac{5}{x+1} \quad (1)$$

$$\frac{2(x-4)(x+2)(x+1)}{x-4}+\frac{3(x-4)(x+2)(x+1)}{x+2}=\frac{5(x-4)(x+2)(x+1)}{x+1} \quad (2)$$

$$2(x+2)(x+1)+3(x-4)(x+1)=5(x-4)(x+2) \quad (3)$$

$$2x^2+6x+4+3x^2-9x-12=5x^2-10x-40 \quad (4)$$

$$7x=-32, \qquad x=-32/7 \quad (5)$$

II

$$\frac{7}{x^2-3x+2}-\frac{4}{x^2-1}=\frac{3}{x^2+3x+2} \quad (1)$$

$$\frac{7}{(x-2)(x-1)}-\frac{4}{(x+1)(x-1)}=\frac{3}{(x+1)(x+2)} \quad (2)$$

$$7(x+1)(x+2)-4(x-2)(x+2)=3(x-2)(x-1) \quad (3)$$

$$7x^2+21x+14-4x^2+16=3x^2-9x+6 \quad (4)$$

$$30x=-24, \quad \text{and} \quad x=-4/5$$

Note the factors that are canceled as you pass from (2) to (3) in II. In the first fraction in (2) the factors $(x-2)$ and $(x-1)$ occur in the denominator. Hence these factors of the l.c.m. are omitted, while the remaining factors $x+1$ and $x+2$ are written in the first term in (3), and so for the other fractions in (2).

EXERCISES

Solve the following equations.

1. $\frac{x+8}{2}-\frac{x-9}{12}+\frac{x-17}{6}=\frac{4x-7}{2}+\frac{x+3}{3}+\frac{5-31x}{12}$

2. $\frac{1}{x-2}+\frac{1}{x-3}=\frac{2}{x-4}$

3. $\frac{1}{x-1}+\frac{1}{x+2}=\frac{4}{2x-1}$

4. $\frac{8}{2x+3}-\frac{1}{x-4}=\frac{3}{x+2}$

5. $\frac{1}{x-1}+\frac{1}{x-2}=\frac{5}{x-3}-\frac{3}{x-4}$

91. ***Exceptional cases.***—To complete the discussion on page 74 we shall consider certain equations containing fractions.

Example 1. Solve $\frac{2}{x+2} + \frac{3}{x-3} = \frac{5}{x+3}$. To clear of fractions we multiply by $(x+2)(x-3)(x+3)$. This multiplier is zero for $x = -2, 3, -3$. But none of these turns up as a root in the solution.

$$\frac{2}{x+2} + \frac{3}{x-3} = \frac{5}{x+3}$$
$$2(x-3)(x+3) + 3(x+2)(x+3) = 5(x+2)(x-3)$$
$$2x^2 - 18 + 3x^2 + 15x + 18 = 5x^2 - 5x - 30$$
$$20x = -30$$
$$x = -3/2$$

Example 2. Solve $\frac{5x^2 - 7x - 6}{x-2} = 4$.

The solution at the right shows 1/5 and 2 as roots. But $x = 2$ cannot be a root of (1) since substituting this value reduces the left member to 0/0, which has no meaning. Hence we conclude that $x = 1/5$ is the only solution of this equation.

$$\frac{5x^2 - 7x - 6}{x-2} = 4 \quad (1)$$
$$5x^2 - 7x - 6 = 4x - 8 \quad (2)$$
$$5x^2 - 11x + 2 = 0 \quad (3)$$
$$(5x-1)(x-2) = 0 \quad (4)$$
$$x = 1/5,\ 2$$

Example 3. Solve $\frac{2}{x-1} + \frac{5}{x+1} = \frac{4}{x^2-1}$. Clearing of fractions we have $2(x+1) + 5(x-1) = 4$, or $x = 1$. But $x = 1$ is not a root of the given equation since $x = 1$ is not a permissible substitution. Hence this equation has no root.

$$\frac{2}{x-1} + \frac{5}{x+1} = \frac{4}{x^2-1}$$
$$2(x+1) + 5(x-1) = 4$$
$$x = 1$$

The general rule is:

If a value of x *obtained by solving a fractional equation reduces a denominator to zero, it is not a root. Otherwise such a value of* x *is a root.*

Hence any value of the unknown obtained by solving a fractional equation must be checked by seeing whether it will reduce a denominator to zero. If it does, this value must be rejected. This checking is usually very simple since it is easily seen whether a given number makes a denominator zero. It can usually be done at a glance.

EXERCISES

Solve the following equations and in each case decide whether the value obtained is actually a root of the given equation. State the reason for your decision.

1. $\dfrac{1}{x+3} - \dfrac{1}{x+2} = \dfrac{7}{x^2+7x+12}$

2. $\dfrac{5}{x-2} + \dfrac{1}{x+2} = \dfrac{15}{x^2-4}$

3. $\dfrac{1}{x^2+x-12} + \dfrac{1}{x^2-x-6} = \dfrac{2}{x^2-2x-3}$

4. $\dfrac{1}{x^2+7x+12} - \dfrac{2}{2x^2+7x-4} = \dfrac{6}{2x^2+5x-3}$

5. $\dfrac{4}{x+5} - \dfrac{1}{x-5} = \dfrac{-10}{x^2-25}$

6. $\dfrac{3}{3x^2-8x-3} - \dfrac{1}{x^2+x-12} = \dfrac{1}{3x^2+13x+4}$

7. $\dfrac{1}{x^2+11x+30} + \dfrac{3}{x^2+3x-18} = \dfrac{4}{x^2+2x-24}$

8. $\dfrac{5x}{x^2-25} - \dfrac{2x}{x^2-7x+10} = \dfrac{3x+7}{x^2+3x-10}$

9. $\dfrac{x+1}{x^2+10x+21} + \dfrac{3x+1}{x^2+8x+15} = \dfrac{4x+5}{x^2+12x+35}$

10. $\dfrac{2x+1}{x^2-4x-21} - \dfrac{x+5}{x^2-5x-14} = \dfrac{x+1}{x^2+5x+6}$

11. $\dfrac{x+4}{x^2-6x-7} + \dfrac{2x+3}{2x^2-13x-7} = \dfrac{4x+1}{2x^2+3x+1}$

12. $\dfrac{3x+1}{3x^2-x-2} + \dfrac{x-3}{x^2-6x+5} = \dfrac{6x-22}{3x^2-13x-10}$

13. $\dfrac{2x+5}{x^2+10x+25} - \dfrac{x+3}{x^2+4x-5} = \dfrac{x+1}{x^2-1}$

14. $\dfrac{4x+1}{x^2-3x+2} - \dfrac{3x+5}{x^2+3x-4} = \dfrac{x+7}{x^2+2x-8}$

15. $\dfrac{x+3}{2x^2-5x+2} + \dfrac{x+5}{x^2+4x-12} = \dfrac{3x-1}{2x^2+11x-6}$

16. $\dfrac{x+2}{x+3} + \dfrac{x-2}{3-x} = \dfrac{5}{x^2-9}$

17. $\dfrac{2x+3}{x-5} + \dfrac{3-x^2}{x^2-6x+5} = \dfrac{x+3}{x-1}$

92. ***Literal equations.***—In an equation several letters may be involved, one of which is regarded as the unknown. The problem in solving such an equation is to find the value of this unknown in terms of the other letters.

$$\begin{aligned} 4x - 2(x - a) &= 2c \\ 4x - 2x + 2a &= 2c \\ 2x &= 2c - 2a \\ x &= c - a \end{aligned} \qquad \text{I}$$

In such equations it is customary to use one of the last letters of the alphabet for the unknown. Thus in I at the right x is used for the unknown. However, such an equation may be solved for any one of its letters "in terms of the other letters."

Check:

$$\begin{aligned} &4(c - a) - 2(c - a - a) \\ &= 4c - 4a - 2c + 4a \\ &= 2c \end{aligned}$$

Example. In II at the right solve equation (1) for R, d, and q.

Equations (2), (4), (5) give the required solutions. In this case all the different letters in (1) are regarded in succession as unknowns.

$$\begin{aligned} D &= dq + R && (1) \\ R &= D - dq && (2) \\ dq &= D - R && (3) \\ d &= \frac{D - R}{q} && (4) \\ q &= \frac{D - R}{d} && (5) \end{aligned} \qquad \text{II}$$

EXERCISES

1. State the meaning of equation (1) in the above Example. (D = dividend, d = divisor, q = quotient, R = remainder.)

2. Translate equation (2) into a rule stated in words.

3. Translate equations (4) and (5) into rules stated in words.

4. Solve $i = prt$ (interest rule) for p, r, t, and state in words the meaning of each solution.

5. Solve $A = \frac{1}{2}(a + b)h$ (area of trapezoid) for h, a, and b.

6. If p, l, w are the perimeter, length, and width of a rectangle, then $p = 2l + 2w$. Solve the equation for l and w.

7. Solve $A = p + prt$ (amount, principal, rate, time) for p, r, and t, and state in words the meaning of each solution.

Solve for x:

8. $ax + bx + cx = 1$

9. $\frac{a}{x} + \frac{b}{x} + \frac{c}{x} = d$

10. $ax + 4b = cx - d$

11. $(a + b)x + (a - b)x = 4ab$

12. $\frac{ax + b}{bx - c} = d$

13. $\frac{ax - a^2}{x + b} = b$

14. $\frac{a + x}{a - x} = \frac{a + b}{a - b}$

15. $\frac{x}{a - 1} + \frac{x}{a + 1} = 6$

16. $\frac{4}{x + a} = \frac{5}{x - a}$

93. Making formulas.—Rules used in mathematics are usually stated as formulas. Many of the rules used in business may also be thus expressed.

Problem. For the fifth zone the parcel post rate is 11 cents for the first pound and 5.3 cents for each additional pound. Express the rule as a formula.

$$c = 11 + 5.3(n - 1)$$

The required formula is given at the right. In this formula n takes only integral values, since a fraction of a pound is counted as a whole pound. Thus the postage on a package weighing 4 lb. 7 oz. is the same as on a package weighing 5 lb.

By giving n the values 1, 2, 3, . . . , in this formula, the postage on any weight package is found.

PROBLEMS

1. Write a formula giving package postage for the third zone, the rate being 9 cents for the first pound and 2 cents for each additional pound.

2. The parcel post rate for the eighth zone is 15 cents for the first pound and 11 cents for each additional pound. State this rule as a formula.

3. A rule for making coffee is: Use 1 spoonful for each cup and 1 spoonful for the pot. Write this rule as a formula for making n cups of coffee.

4. The cost of a ten-word telegram between two cities is 62 cents and 3.5 cents for each additional word. Write a formula giving the cost of an n-word message. Note that this formula does not hold for n less than 10.

5. A taxi company charges 20 cents for the first quarter of a mile and 5 cents for each additional quarter of a mile. Using the quarter of a mile as the unit of distance, write a formula giving the charge for n quarters of a mile.

6. Let F° represent the temperature-reading in degrees on an ordinary Fahrenheit thermometer and C° the reading on a centigrade thermometer. If we subtract 32° from the Fahrenheit reading and multiply the remainder by 5/9, the result will be the centigrade reading. Express this relation as a formula.

7. If A represents the "amount" in one year of an investment, p the principal invested, and r the rate of interest, write a formula giving A in terms of p and r.

8. The total surface of a circular cylinder is obtained by taking twice the area of one end and adding the curved area (the lateral area). The lateral area is equal to the length (or height) of the cylinder multiplied by the circumference. Using A for total area, r for radius, h for height, write a formula giving the total area.

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In such equations it is customary to use one of the last letters of the alphabet for the unknown. Thus in I at the right x is used for the unknown. However, such an equation may be solved for any one of its letters "in terms of the other letters."

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Check:

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EXERCISES

1. State the meaning of equation (1) in the above Example. (D = dividend, d = divisor, q = quotient, R = remainder.)
2. Translate equation (2) into a rule stated in words.
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6. If p, l, w are the perimeter, length, and width of a rectangle, then $p = 2l + 2w$. Solve the equation for l and w.
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Solve for x:

8. $ax + bx + cx = 1$
9. $\frac{a}{x} + \frac{b}{x} + \frac{c}{x} = d$
10. $ax + 4b = cx - d$
11. $(a + b)x + (a - b)x = 4ab$
12. $\frac{ax + b}{bx - c} = d$
13. $\frac{ax - a^2}{x + b} = b$
14. $\frac{a + x}{a - x} = \frac{a + b}{a - b}$
15. $\frac{x}{a - 1} + \frac{x}{a + 1} = 6$
16. $\frac{4}{x + a} = \frac{5}{x - a}$

93. Making formulas.—Rules used in mathematics are usually stated as formulas. Many of the rules used in business may also be thus expressed.

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The required formula is given at the right. In this formula n takes only integral values, since a fraction of a pound is counted as a whole pound. Thus the postage on a package weighing 4 lb. 7 oz. is the same as on a package weighing 5 lb.

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By giving n the values 1, 2, 3, . . . , in this formula, the postage on any weight package is found.

PROBLEMS

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94. Solving formulas.—At the right, formula (1) is solved for r. Explain each step in this solution. Very likely you have no idea of the meaning of this formula and of the letters used in it. But this does not prevent you from solving the formula for any of these letters. It has a very definite meaning in the science of electricity.

$$C = \frac{nE}{nr + R} \qquad (1)$$
$$Cnr + CR = nE \qquad (2)$$
$$Cnr = nE - CR \qquad (3)$$
$$r = \frac{nE - CR}{Cn} \qquad (4)$$
$$r = \frac{nE}{Cn} - \frac{CR}{Cn} \qquad (5)$$
$$r = \frac{E}{C} - \frac{R}{n} \qquad (6)$$

By starting with (1) we discover by a simple algebraic process an important fact about electricity represented by (6). It is quite thinkable that an electrician may know the fact represented by (1), but if he is ignorant of algebra he will not be able to discover (6), though the fact represented by it may be of practical importance to him.

This is an example of a rôle that algebra is made to play very extensively in the sciences and in their practical applications. The important point about this example is that if the fact about electric current represented by any one of these equations is known, then the fact represented by any of the other equations can be discovered by this simple mathematical process. Without it each of these facts would have to be discovered by experimentation which might be quite laborious. People who work on physical problems make very extensive use of mathematics.

The process illustrated above is sometimes called changing the "subject" of a formula. In (1) above, C is the "subject" of the sentence represented by the formula, while in (6), r is the subject.

EXERCISES

1. Solve equation (1) above successively for n, E, and R.
2. Solve $S = \frac{n}{2}(a + l)$ successively for n, a, l.
3. Solve $l = a + (n - 1)d$ for a, n, d.
4. If the dimensions of a rectangular solid are l, w, h, and its total surface is S, write a formula giving S in terms of l, w, h.
5. Solve the formula obtained in exercise 4 for l, w, h.
6. Write a formula giving the volume V of a rectangular solid in terms of l, w, h (see exercise 4). Solve this formula for l, w, and h.

7. Solve $ax + 3b = cx + d$ for x.
8. Solve $l(W + W_1) = l_1w_1$ for each of the letters in terms of the others.
9. Solve $(a - b)x - (a + b)x + 4a^2 = 0$ for x.
10. Solve $(v - n)d = (v - n_1)d_1$ for each letter in terms of the others.

Note that n and n_1 are different, as are also d and d_1.

11. Solve $S = \frac{a - al}{l - r}$ for each of the letters in terms of the others.
12. Solve $F = k\frac{mm_1}{r^2}$ for k, m, and m_1.
13. Solve $S = \frac{a}{l - r}$ for a and r.
14. Solve $l = \pi s(r + r_1)$ for s, r, and r_1.
15. Solve $\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$ for a, b, and f.
16. Solve $i = \frac{E}{R + \frac{r}{2}}$ for E, R, and r.

17. Solve $R = \frac{r_1r_2}{r_1 + r_2}$ for r_1 and r_2.
18. Solve $R = \frac{2as}{l - 2s}$ for a, l, and s.
19. Solve $E = \frac{f}{p + x}$ for f, p, and x.
20. Solve $F = \frac{W}{2R}(R - r)$ for W, R, and r.
21. Solve $s = v_0t + \frac{1}{2}gt^2$ for v_0 and for g.
22. Solve $i = \frac{d}{1 - d}$ for d, and $d = \frac{i}{1 + i}$ for i.
23. Solve $D = \frac{5}{9}(F - 32)$ for F.
24. Solve $H = \frac{2ab}{a + b}$ for a and b.
25. Solve $A = P(1 + prt)$ for P, p, r, and t.
26. Solve $1 - di = d$ for d and i.
27. Solve $C = \frac{Kab}{a + b}$, (1) for K, (2) for a, (3) for b.
28. Solve $S = \frac{lr - a}{r - 1}$ (1) for r, (2) for l, (3) for a.
29. Solve $A = 2\pi r^2 + 2\pi rb$, for b. Can you solve this equation for r?

95. Solution of problems.—The equation is the most effective instrument for solving problems that has ever been invented. At our present stage we can use it in solving only simple types of problems, and not a few of these must be in the nature of mere puzzles. But as we go on with our work of accumulating algebraic equipment, we shall be making our problems increasingly genuine. The fact is that the genuine problems in this world are often very difficult and we are compelled to learn to use our tools on those that have been simplified artificially.

Problem. If 7 is added to a number and the sum multiplied by 5, the product is 45 more than 3 times the number. Find the number.

The equation at the right is an exact translation of this problem into the language of algebra. The required number will be found by solving the equation.

$$5(x + 7) = 3x + 45$$

Solve this equation, and check by showing that the answer satisfies the conditions stated in the problem. It is not sufficient to check by substituting in the equation. That will merely check the work of solving the equation, but will not check the correctness of the equation itself.

PROBLEMS

1. There are 72 pupils in two classes. If 7 are taken from the first class and put into the second, then the first has 2 less than the second. How many are there in each class?

2. There are a pupils in two classes. If b are taken from the first class and put into the second, there are c more in the first than in the second. Find the number in each class. Solve problem 1 by substituting in the answer to this one.

3. Make a problem like problem 1 and find the answer by using the result obtained in problem 2.

4. If a is added to a number and the product multiplied by b, the product is c more than d times the number. What is the number? Solve the Problem above by using this formula.

96. Problems containing consecutive integers.—If n is an integer, then a series of consecutive integers beginning with n is represented by n, $n + 1$, $n + 2$, $n + 3$, ... ; a series of even or odd integers beginning with n is represented by n, $n + 2$, $n + 4$, $n + 6$, . . . , n itself in this case being even if the integers are even and odd if the integers are odd.

PROBLEMS

1. Find 4 consecutive integers whose sum is 198.
2. Find 3 consecutive even integers whose sum is 108.
3. Find 5 consecutive odd integers whose sum is 145.
4. Find 3 consecutive integers such that 4 times the first minus twice the second plus 3 times the third equals 79.
5. Find 3 consecutive integers such that half the first plus the second minus one-third the last equals 26.
6. Find 4 consecutive even integers such that one-half the first minus one-third the second plus one-fourth the third plus twice the fourth equals 22.
7. Of 3 numbers the second is 12 greater than the first and the third is 9 greater than the second. One-sixth the first number minus one-eighth the second plus one-third the last number equals 10. What are the numbers?
8. Find 4 consecutive integers whose sum is a. Find three different values of a that make this problem possible. Note that the answers must be integers.
9. Find 3 consecutive odd integers whose sum is a. Find two values of a that make this problem possible. What kind of a number must a be?
10. Of 4 numbers the second is a greater than the first, the third a greater than the second, and the fourth a greater than the third. Their sum is s. Find the numbers. Find a value of a and of s that makes these numbers integers.
11. Find 4 consecutive integers such that 6 times the second less 4 times the third equals a. Find three values of a that make this problem possible.

97. ***Problems involving uniform motion; the "courier" problem.***—An object is said to be moving uniformly if, in the whole course of the motion, it passes over equal distances in equal intervals of time. Thus a train may be going at a uniform (nearly) speed of 50 miles an hour. Light is supposed to go at a uniform speed when passing through vacant space. Sound under certain circumstances travels at a uniform speed. Many speeds are very nearly uniform, and it is such speeds that we shall use in our problems.

Speed is given at so many units of distance in a unit of time. A car may be going 50 miles per hour, sound may travel 1100 feet per second. To find the distance d which a body moving at a uniform speed s goes in t units of time, we use the formula $st = d$. (speed × time = distance.)

$$st = d$$
$$s = d/t$$
$$t = d/s$$

Problem. (An ancient problem) At five o'clock in the morning a king sent a messenger (courier) traveling 6 miles per hour with orders for his army. At eight o'clock he sent another messenger traveling 9 miles per hour with different orders to overtake the first. When will the second messenger overtake the first?

$$\begin{aligned} 6t &= 9(t-3) \\ 6t &= 9t - 27 \\ -3t &= -27 \\ t &= 9 \end{aligned}$$

SOLUTION: Let t be the time (number of hours) in which the event will take place. Then $6t = 9(t-3)$. The solution is shown at the right.

PROBLEMS

1. A car traveling 55 miles per hour sets out to overtake a car that left 1½ hours earlier going at the rate of 45 miles per hour. In how long after it starts will the faster car overtake the slower?

2. An airplane going 120 miles per hour is followed by one going 175 miles per hour. If the slower plane has 2¾ hours start, how long after it starts will it be overtaken?

3. A ship going from New York to England at the rate of 14 knots (nautical miles per hour) is 960 nautical miles out when a very fast ship going 30.5 knots starts from New York along the same route. In how many hours after the second ship starts will the ships be together? Show that the equation at the right will give the correct answer.

$$30.5t = 14t + 960$$

4. Two objects moving at speeds s_1 and s_2 start at the same time, s_1 being the greater speed. In how many units of time will the faster object gain a distance a on the slower? Solve problem 3 by substituting in the formula thus obtained.

5. In the Problem above, how many miles must the second messenger gain on the first before overtaking him? Solve this problem by using the formula obtained in problem 4.

6. Solve problems 1 and 2 by substituting in the formula obtained under problem 4. What quantities must be found before substituting in the formula?

7. Two airplanes traveling on an east-west route have speeds of 140 miles per hour and 185 miles per hour respectively. If they start from the same airport at the same time, in how many hours will they be 375 miles apart (a) if they go in the same direction? (b) if they go in opposite directions?

8. In how many minutes after 4 o'clock will the hands of a clock be together?

SUGGESTION: Let a minute be the unit of time and a minute space on the clock face be the unit of distance. Then the speeds are 1 and 1/12. The distance to be gained is 20.

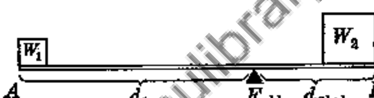
9. At what time after 6 o'clock will the hands of the clock be at right angles? (15 minute-spaces apart.) Find two answers between 6 o'clock and 7 o'clock.

10. In how many minutes will the minute hand gain n minute-spaces on the hour hand? Solve problems 8 and 9 by substituting in this formula.

11. An airplane going 125 miles per hour left the airport 2½ hours ahead of a fast plane starting to overtake it. How fast will the second plane have to go to overtake the first in 6½ hours?

98. *Problems involving the simple lever.*—The lever is one of the oldest and most important of the contrivances that we use in our machinery. Its fundamental principle is very simple.

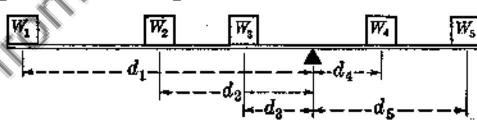
The lever consists of a straight bar resting on a point F called the fulcrum. (See the figure.) The parts AF and FB are the arms of the lever. In the figure, the lengths of these are represented by d_1 and d_2. At A the lever carries a weight w_1 and at B a weight w_2.



That is, the weights w_1 and w_2 are at distances d_1 and d_2 from the fulcrum and on opposite sides of it. In order that the lever shall balance we must have $d_1w_1 = d_2w_2$. This is the principle of the lever.

$$d_1w_1 = d_2w_2 \quad \text{(I)}$$

The teeter board is a primitive example of this kind of a lever. As suggested in the next figure, the relation may be complicated by placing several weights on the lever. In order that the lever shall balance we must have the relation shown below the figure.



$$d_1w_1 + d_2w_2 + d_3w_3 = d_4w_4 + d_5w_5 \quad \text{(II)}$$

Clearly, if any three of the quantities d_1, d_2, w_1, w_2 in I are given, the fourth may be found from $d_1w_1 = d_2w_2$. Again, if any 9 of the 10 quantities in $d_1w_1 + d_2w_2 + d_3w_3 = d_4w_4 + d_5w_5$ are given, the one remaining may be found by using this equation.

If the distances on one side of the fulcrum are regarded as positive and on the other side as negative, then the condition that the lever shall balance is $d_1w_1 + d_2w_2 + d_3w_3 + \ldots = 0$.

PROBLEMS

1. If in equation I, page 87, $d_1 = 7$, $d_2 = 9$ and $w_1 = 17$, find w_2.
2. If in the same equation $w_1 = 42$, $w_2 = 56$, and $d_2 = 8$, find d_1.
3. If the length of the lever AB is 12, and $w_1 = 84$ and $w_2 = 108$, find AF and FB. (d_1 and d_2.)
4. If in II, $w_1 = 8$, $w_2 = 6$, $w_3 = 4$, $w_4 = 12$, $w_5 = 20$, $d_1 = 12$, $d_2 = 7$, $d_3 = 3$, $d_4 = 6$, find d_5.
5. A lever 14 feet long carrying weights of 52 and 96 at its ends rests on a fulcrum at its middle point. What additional weight must be placed 4½ feet from the fulcrum to make the lever balance? Draw a figure.
6. A lever 16 feet long carries weights of 18 and 42 pounds at its ends, a weight of 24 pounds 3 feet from the 42-pound weight. At what point must the fulcrum be placed to make the lever balance?
7. A lever 15 feet long carries a 56-pound weight at one end and a 24-pound weight 2 feet from this end. At the other end it carries a 42-pound weight and 3 feet from this end a 20-pound weight. At what point must the fulcrum be placed to make the lever balance?

99. ***Problems involving the values of coins; work problems.***—The following illustrates the first of these.

Problem. A collection of 35 dimes and quarters is worth \$6.05. How many coins of each kind are there? Show that the equation at the right represents the problem.

$$x = \text{no. of dimes}$$
$$35 - x = \text{no. of quarters}$$
$$10x + 25(35 - x) = 605$$

PROBLEMS

1. Solve the equation in the above Problem and check by finding whether the answer satisfies the problem.
2. A collection of dimes and half dollars is worth \$12.60. How many coins of each kind are there if there are 50 coins in the collection?
3. In a collection of nickels, dimes, and quarters there are 14 more nickels than dimes and the value of the collection is \$8.55. The total number of coins is 86. How many of each kind are there?
4. In a collection of coins there are 8 more nickels than dimes and 78 more pennies than nickels. How many of each kind are there if the total value is \$13.10?
5. In a collection of pennies, nickels, and dimes, the nickels are worth 33 cents more than the pennies, and the dimes are worth 10 cents less than the nickels. How many coins of each kind are there if the value of the whole collection is \$1.67?

MISCELLANEOUS PROBLEMS

1. Two numbers are in the ratio 3 to 7 and their sum is 80. What are the numbers? SUGGESTION: Let $3x$ and $7x$ represent the numbers.

2. The ratio of two numbers is 3 to 5. If 6 is added to each number, their sum will be 76. Find the numbers.

3. The ratio of two numbers is 4 to 7. If 12 is added to each number, the ratio of the sums will be 5 to 8. Show that the equation at the right represents this problem. Then solve the equation and check the answer.

$$8(4x + 12) = 5(7x + 12)$$

4. Divide the number 96 into three parts, such that the second part is twice the first, and the third part is 30 less than twice the second.

5. In a collection of 140 coins there are nickels, dimes, and quarters. There are 28 more dimes than quarters, and twice as many nickels as dimes. How many coins of each kind are there?

6. Divide the number 132 into three parts such that the parts will be in the same ratio as the numbers 3, 7, 12. SUGGESTION: Let $3x$, $7x$, $12x$ represent the numbers.

7. A train goes 3 hours at a certain rate and then goes 4 hours at a rate 8 miles per hour greater, covering in all 420 miles. Find the two rates.

8. In two rectangles with the same width, the length of one is 12 feet greater than the width, and the length of the other is 4 less than twice its width. Find the dimensions of the rectangles if the sum of their perimeters is 176 feet.

9. A man had to go 650 miles to reach his home. He flew part of the way at a rate of 140 miles per hour and drove the rest of the way at 45 miles per hour, making the total distance in 6 hours. How far did he fly?

10. Find three consecutive integers such that 4 times the first less 3 times the second plus twice the last is 52.

11. The ratio of two numbers is 3 to 7. If 6 is subtracted from both numbers the ratio is 1 to 5. Find the numbers.

12. The ratio of two numbers is a to b. If c is subtracted from both numbers, the ratio is m to n. Find the numbers.

13. Solve problems 3 and 11 by substituting in the solution found in problem 12.

14. Two cities A and B are 850 miles apart. A plane starts from A going toward B at 140 miles per hour. At the same time another plane starts from B going toward A. How fast does the second plane have to go if they meet in $2\frac{3}{4}$ hours?

15. The arms of a lever are 8 feet and 5 feet respectively. At the end of the shorter arm it carries a weight of 37 pounds. What must be the weight at the other end to make the lever balance? Use positive and negative numbers to represent the lever arms.

PROBLEMS

1. Divide the number 152 into two parts which will be in the ratio 7:12.

2. Obtain a general formula for solving problems like the preceding by dividing the number c into two parts which will be in the ratio $a:b$.

3. A can do a piece of work in 11 hours and B can do it in 12½ hours; in how many hours can they do it when working together?

Suggestion: Let 1 represent the whole piece of work. Then the work that can be done by A in one day is $\frac{1}{11}$ and the work that can be done by B is $\frac{1}{12\frac{1}{2}}$. Hence the work that both can do in one day is $\frac{1}{11} + \frac{1}{12\frac{1}{2}}$. Hence the time that both require is $1 \div \left(\frac{1}{11} + \frac{1}{12\frac{1}{2}}\right)$. Also solve this problem by substituting in the formula that you obtain by solving the next problem.

4. Obtain a general formula for solving problems like the preceding by supposing that A can do the work in a hours and B in b hours.

5. A bath has two taps, one of which delivers a gallons in p minutes and the other b gallons in q minutes; find a formula showing how many gallons the two taps deliver in t minutes.

6. One pipe delivering water into a reservoir can fill it in a days. Of two pipes taking water out of it, one can empty it in b days and the other can empty it in c days. If the reservoir is full, in how many days will it be empty if all three pipes are running? Under what condition is this problem possible?

7. If the reservoir in the preceding problem is empty when the three pipes begin operating, in how many days will the reservoir be full? Under what condition is this problem possible?

8. A certain sum of money will pay A's wages for a days and the same sum will pay B's wages for b days; find a formula showing the number of days' wages of both men that this sum will pay. Reduce the expression to a simple fraction.

9. In how many minutes after 7 o'clock will the hands of a clock be at right angles?

10. A straight line is drawn from 10 to 4 on the face of a clock; in how many minutes after 6 o'clock will the hands of the clock make equal angles with this line?

11. Find two consecutive integers such that the difference of their squares is 71.

12. Prove that the difference of the squares of two consecutive integers is always an odd number.

13. Two integers differ by 3; show that the difference of their squares is 3 greater than some multiple of 6.

CHAPTER 8:

LINEAR EQUATIONS AND THEIR SOLUTION

In an equation of the type solved thus far there is one unknown, and the equation is sufficient to determine the value of this unknown. By solving the equation this value is found. We are now to study equations containing more than one unknown.

100. Linear equations in two unknowns.—An equation of the type $ax + by + c = 0$ is called a linear equation in two unknowns. The letters a, b, c represent fixed numbers, and x and y represent unknown quantities. The letters x and y are called the unknowns. Each of the equations shown at the right is of this type.

$$\begin{aligned} ax + by + c &= 0 \\ 2x + y - 12 &= 0 \\ 3x - 4y + 12 &= 0 \\ -5x + 7y + 16 &= 0 \end{aligned}$$

If in $ax + by + c = 0$, $a = 2$, $b = 1$, $c = -12$, we have $2x + y - 12 = 0$, or $2x + y = 12$. If we write this equation in the form $y = 12 - 2x$, we see that any value whatever may be given to x and a corresponding value found for y. In this way we may find an endless (infinite) number of pairs of values of x and y that satisfy this equation. Among these are the following.

$$\begin{aligned} 2x + y - 12 &= 0 \\ y &= 12 - 2x \end{aligned}$$

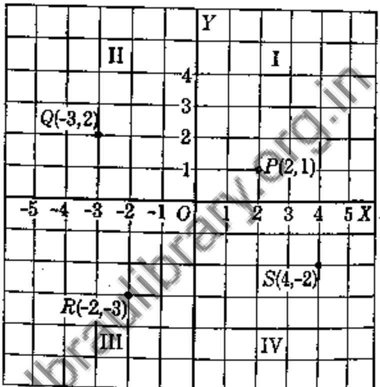
$$\begin{cases} x = 0 \\ y = 12 \end{cases} \begin{cases} x = 1 \\ y = 10 \end{cases} \begin{cases} x = 2 \\ y = 8 \end{cases} \begin{cases} x = 3 \\ y = 6 \end{cases} \begin{cases} x = 4 \\ y = 4 \end{cases} \begin{cases} x = 5 \\ y = 2 \end{cases} \begin{cases} x = 6 \\ y = 0 \end{cases} \begin{cases} x = 7 \\ y = -2 \end{cases}$$

In general, for any linear equation in two unknowns there is an infinite number of pairs of numbers that satisfy it. The unknowns in such an equation are also called variables, since each of them may represent any one of a set of numbers. That is, each of them may "vary" over a set of values.

We shall now show how a linear equation in two unknowns may be represented by a straight line.

In the equation $y = 12 - 2x$, x is regarded as the *independent* variable and y as the *dependent variable.*

101. A system of coordinates.—To construct a system of coordinates, called the Cartesian system, draw two lines at right angles meeting at a point O, which we shall call the origin. Mark the horizontal line X and the vertical line Y as shown in the figure. The lines OX and OY are called respectively the x-axis and y-axis. On these axes lay off units forming scales of signed numbers. The x-axis and the y-axis taken together are called the coordinate axes.



The location of a point in this plane may be described by giving its distance and direction from each of the coordinate axes. As shown in the figure, distances to the right of the y-axis are regarded as positive, and to the left negative; while distances upward from the x-axis are regarded as positive and downward negative.

The number indicating the distance and direction of a point from the y-axis (distance parallel to the x-axis) is called the x-coordinate, or abscissa, of the point; while the number indicating its distance and direction from the x-axis is called the y-coordinate, or the ordinate, of the point. These numbers taken together are the coordinates of the point. In representing a point by its coordinates, the x-coordinate is written first. Thus the point P in the figure is represented by (2,1) or by $P(2,1)$. The points Q, R, S are represented by $Q(-3,2)$, $R(-2,-3)$, $S(4,-2)$; or simply by $(-3,2)$, $(-2,-3)$, $(4,-2)$. The origin is the point $O(0,0)$ or simply $(0,0)$.

The parts into which the coordinate axes divide the plane are called quadrants, which are numbered as shown by the Roman numerals in the above figure. The point P is in the first quadrant, Q in the second, R in the third, and S in the fourth.

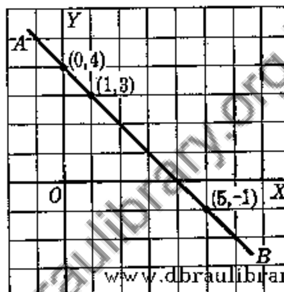
For the point P, $x = 2$, $y = 1$. We speak of P as "the point $x = 2$, $y = 1$." Similarly, Q is the point $x = -3$, $y = 2$; R is the point $x = -2$, $y = -3$; and S is the point $x = 4$, $y = -2$.

The plane with its points thus described is also called the XY-plane.

102. Graph of the equation $x+y=4$.—The equation $x+y=4$ is satisfied by these pairs of numbers:

$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=-1$	$x=5$	$x=-2$	$x=6$
$y=4$	$y=3$	$y=2$	$y=1$	$y=0$	$y=5$	$y=-1$	$y=6$	$y=-2$

The points $x=0$, $y=4$; $x=1$, $y=3$; etc., are shown in the figure. It is easily seen that these points are in a straight line.



If $x=\frac{1}{2}$, then $y=3\frac{1}{2}$, and the point $x=\frac{1}{2}$, $y=3\frac{1}{2}$ also lies on this same line. In general: (1) the coordinates of any points on the line AB satisfy the equation $x+y=4$, and (2) any pair of numbers that satisfy $x+y=4$ are the coordinates of a point on this line. Because of these facts we say that the equation $x+y=4$ is the equation of the line AB and that the line AB is the graph of the equation $x+y=4$.

We can see as follows that this graph is a straight line. As x is increased by 1, y is decreased by 1; or, in general, as x is increased by any given amount, then y is decreased by the same amount. Hence the direction from point to point remains the same. But a point that moves in the same direction moves in a straight line.

EXERCISES

1. Find a number of pairs of values of x and y that satisfy the equation $x-y=4$. Locate the point in the plane determined by each of these pairs of numbers. How are these points related? What is the graph of the equation $x-y=4$?

2. Find a number of pairs of numbers that satisfy $2x+3y=6$. Try to get integral numbers. Locate the points determined by your pairs of numbers. How are these points related? What is the graph of $2x+3y=6$?

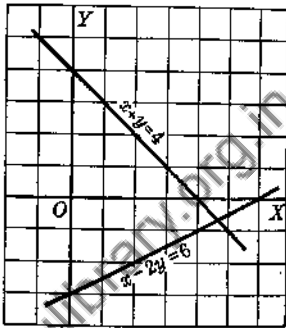
3. Try to construct the graph of $3x-5y=10$. Is this graph a straight line?

4. Construct the graphs of $x+y=2$, $x+y=3$, $x+y=5$. How are these related to the graph of $x+y=4$?

5. Construct the graphs of $x-y=1$, $x-y=2$, $x-y=3$, $x-y=5$. How are these related to the graph of $x-y=4$?

105. The intersection point of the graphs of two linear equations.—Consider the equations at the right. The graphs are shown below the equations.

$$x + y = 4$$
$$x - 2y = 6$$



The intersection point lies on both lines, and hence its coordinates must satisfy the equation $x + y = 4$ and also $x - 2y = 6$. Since the two lines meet in only one point, it is apparent that only one pair of numbers will satisfy both equations.

From the graph it appears that the coordinates of the intersection point are about $x = 4\frac{1}{2}$, $y = -\frac{1}{2}$. The attempted check given at the right shows that this result is not quite correct. As we shall see on page 97, the exact answer is $x = 4\frac{2}{3}$, $y = -\frac{2}{3}$.

$$4\tfrac{1}{2} + (-\tfrac{1}{2}) = 4$$
$$4\tfrac{1}{2} - 2(-\tfrac{1}{2}) = 5\tfrac{1}{2}$$

By using a larger graph the correct results may be approximated more closely. In many simple cases the exact answer may be found.

EXERCISES

Construct the graphs of the following pairs of equations and find approximately the coordinates of the intersection points. By substitution in the original equations check the closeness of your approximations.

1. $x + y = 4$
   $x - y = 4$
2. $x + y = 5$
   $x - 2y = 4$
3. $x - y = -3$
   $x + 2y = 6$
4. $2x - y = 4$
   $x + 3y = 4$
5. $x + 2y = 3$
   $3x - y = 4$
6. $2x - 3y = 6$
   $2x + 3y = 6$
7. $5x - 3y = 15$
   $3x + 5y = 15$
8. $3x + 2y + 6 = 0$
   $2x - 3y - 12 = 0$
9. $6x - 5y = 30$
   $x - 4y = 8$
10. $2x - 5y = 10$
    $10x - 5y = 20$
11. $3x - 8y = 24$
    $5x - y = 5$
12. $7y - 3x + 21 = 0$
    $2y + 5x - 10 = 0$

106. Solution of simultaneous linear equations; method of comparison.—For the present we shall consider two linear equations in two unknowns (see page 91, §100). Solving a pair of such equations consists in finding values of x and y that will satisfy both equations. We shall illustrate by means of an example:

Example. Solve equations (1) and (2) at the right.

We shall find a value of x which when substituted in both equations gives the same value of y.

STEP 1. Solve (1) and (2) for y, obtaining (3) and (4).

STEP 2. To find the value of x that will make these two values of y equal, equate the right members of (3) and (4), obtaining (5).

STEP 3. Solve (5), finding $x = 4\frac{2}{3}$.

STEP 4. Substitute $x = 4\frac{2}{3}$ in (3) and find $y = -\frac{2}{3}$.

$$x + y = 4 \quad (1)$$
$$x - 2y = 6 \quad (2)$$
$$y = 4 - x \quad (3)$$
$$y = \frac{x-6}{2} \quad (4)$$
$$4 - x = \frac{x-6}{2} \quad (5)$$
$$x = 4\tfrac{2}{3} \quad (6)$$
$$y = 4 - (4\tfrac{2}{3}) = -\tfrac{2}{3} \quad (7)$$
$$x = 4\tfrac{2}{3}, \quad y = -\tfrac{2}{3} \quad (8)$$

Check:

$$4\tfrac{2}{3} + (-\tfrac{2}{3}) = 4 \quad (1)$$
$$4\tfrac{2}{3} - 2(-\tfrac{2}{3}) = 6 \quad (2)$$

To check, substitute the values in (8) in both (1) and (2). This method of solving simultaneous equations is called the method of comparison. The values of y in terms of x are "compared" as in (5).

EXERCISES

1–12. Using the method of comparison, solve the pairs of equations given on page 96. Compare the results with those obtained from the graphs.

Solve the following sets of equations. Check answers by substituting in the original equations or by going over the work again with great care.

13. $3x - 5y + 8 = 0$
    $x + 8y - 10 = 0$

14. $2y - 6x + 9 = 0$
    $4x + y - 6 = 0$

15. $5x + 3y = 28$
    $3x + 5y = 36$

16. $\frac{1}{2}x + \frac{1}{3}y = 3$
    $2x - 3y = -14$

17. $\frac{1}{4}x + \frac{1}{4}y = 5$
    $\frac{1}{2}x - \frac{1}{6}y = 1$

18. $-8x + 3y - 12 = 0$
    $3x + 2y - 33 = 0$

19. $3x - 2y = 8$
    $x + 2y = 8$

20. $x + 3y = 8$
    $3x - y = 14$

107. *Solving simultaneous equations; method of addition and subtraction.* —The method of solving simultaneous equations that is usually most easily carried out is the one we are now to study. However, the reason for it is not understood quite so easily as the method of comparison.

Examples will illustrate the method:

Example 1. Solve equations (1) and (2) at the right.

$$\begin{array}{rcll} x + y &=& 4 & (1)\\ x - 2y &=& 6 & (2)\\ \hline 3y &=& -2 & (3)\\ y &=& -\tfrac{2}{3} & (4)\\ x = 4\tfrac{2}{3}, \quad y &=& -\tfrac{2}{3} & (5) \end{array}$$

These are the equations already solved by graphing and by comparison.

STEP 1. Subtract the members of (2) from the members of (1), obtaining $3y = -2$, $y = -\tfrac{2}{3}$.

STEP 2. Substitute $y = -\tfrac{2}{3}$ in (1) and complete the solution.

STEP 3. Check as in §106.

Example 2. Solve $2x - 3y = 15$, $3x + 5y = 40$. The solution is shown at the right.

$$\begin{array}{rcll} 2x - 3y &=& 15 & (1)\\ 3x + 5y &=& 40 & (2)\\ \\ 6x - 9y &=& 45 & (3)\\ 6x + 10y &=& 80 & (4)\\ \hline -19y &=& -35 & \\ y &=& \dfrac{35}{19} & \\ \hline 10x - 15y &=& 75 & (5)\\ 9x + 15y &=& 120 & (6)\\ \hline 19x &=& 195 & \\ & & 195 & \\ x &=& \dfrac{195}{19} & \end{array}$$

STEP 1. Multiply (1) by 3 and (2) by 2. This gives equations (3) and (4).

STEP 2. Subtract (4) from (3) and solve for y, obtaining $y = 35/19$.

STEP 3. Multiply (1) by 5 and (2) by 3, obtaining (5) and (6).

STEP 4. Add (5) and (6) and solve for x. The solution is $x = 195/19$, $y = 35/19$.

In solving for y we first "eliminate" x by multiplying (1) by 3 and (2) by 2 and subtracting. In solving for x we eliminate y by multiplying (1) by 5 and (2) by 3 and adding.

To eliminate x in two equations (1) and (2), multiply (1) and (2) by numbers that will make the coefficients of x in the two equations the same in absolute value. Then add or subtract as may be necessary to eliminate x, and solve for y.

If the value of y is a small integral number, substitute its value in one of the given equations and solve for x. Otherwise, eliminate y by the process used in eliminating x and then solve for x.

EXERCISES

Solve the following pairs of equations.

1. $3y - 4x = 12$
$y - 5x = 3$
2. $7x + 8b = 36$
$11x + 5b = -4$
3. $y = 4 - x$
$3x + 4y = 4$
4. $16m + 3n = 11$
$20m = 30 - 7n$
5. $15c = 8 - 4d$
$12d = 20c - 15$
6. $4x - y = 7$
$4x = 8 + 2y$
7. $2x + 3y + 13 = 0$
$3x - 4y - 6 = 0$
8. $5a + 3b = -9$
$3a = 4b - 17$
9. $m + 2n + 3 = 0$
$3m - n - 5 = 0$

108. Solving simultaneous equations; method of substitution.—It often happens that one of the two given equations is very simple while the other is somewhat complicated. An example is shown at the right. In this case we simplify (2), obtaining (4). Then solve (1) for y and substitute in (4). On solving, this gives $x = 3$. Substituting in (1) gives $y = 1$. This process is called solving by <u>substitution</u>.

$$\begin{aligned} x + y &= 4 \qquad (1) \\ \frac{3x - 5y}{4} + 8x &= 25 \qquad (2) \\ 3x - 5y + 32x &= 100 \qquad (3) \\ 35x - 5y &= 100 \qquad (4) \\ y &= 4 - x \\ 35x - 5(4 - x) &= 100 \\ 35x - 20 + 5x &= 100 \\ 40x &= 120 \\ x &= 3 \end{aligned}$$

EXERCISES

Solve by the method that appears to be the simplest.

1. $y = 3x$
$x + \frac{y}{3} = 34$
2. $\frac{x}{7} + \frac{y}{9} = 10$
$x + \frac{y}{3} = 50$
3. $\frac{x}{2} - \frac{y}{5} = 4$
$\frac{x}{7} + \frac{y}{15} = 3$
4. $3y - 7x = 0$
$\frac{2y}{7} + \frac{5x}{3} = 7$
5. $3x + \frac{1}{2}y = 17$
$\frac{x}{5} - \frac{y}{4} = 0$
6. $\frac{2x - 1}{3} - \frac{y}{2} = 8$
$3x - y = 6$
7. $2x + y = 0$
$\frac{1}{3}x - \frac{3}{4}y = 6$
8. $\frac{6x - 1}{2} - \frac{2y + 1}{3} = 0$
$7x - 3y = 5$
9. $5x - 3 + \frac{y}{6} = 2$
$\frac{2x}{3} + 4 - \frac{y}{2} = 4$
10. $\frac{2x}{3} + \frac{3y}{4} = x - \frac{5y}{6} + 1 = 0$
11. $\frac{5x}{6} + y - 1 = 2x + \frac{3y}{4} + 2 = 0$

109. General solution of two simultaneous linear equations.—To obtain a general formula for solving such equations write the equations in the form (1) and (2) at the right. Note that in each equation a is the coefficient of x and that b is the coefficient of y, while the constant term is c. The subscripts $_1$ indicate the first equation and the subscripts $_2$ indicate the second equation.

$$a_1x + b_1y = c_1 \qquad (1)$$
$$a_2x + b_2y = c_2 \qquad (2)$$
$$a_1b_2x + b_1b_2y = c_1b_2 \qquad (3)$$
$$a_2b_1x + b_1b_2y = c_2b_1 \qquad (4)$$
$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1 \qquad (5)$$
$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad (6)$$
$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (7)$$

The steps in solving $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are shown at the right. Multiplying (1) by b_2 and (2) by b_1 and subtracting leads to the value of x in (6).

To eliminate x multiply (1) by a_2 and (2) by a_1 and subtract. By dividing, this leads to the value of y in (7).

These values of x and y constitute formulas, by substitution in which the solution of any two linear equations may be found.

The substitution to find the values of x and y in $2x + 3y = 5$, $4x + 7y = 9$ is shown at the right above.

$$2x + 3y = 5$$
$$4x + 7y = 9$$
$$x = \frac{5 \cdot 7 - 9 \cdot 3}{2 \cdot 7 - 4 \cdot 3} = 4$$
$$y = \frac{2 \cdot 9 - 4 \cdot 5}{2 \cdot 7 - 4 \cdot 3} = -1$$

We should note the special value of using literal coefficients and obtaining the solution in terms of them as in the first solution above. These literal coefficients enter into the final solution, retaining their identity so that a general formula is obtained for solving all equations of this type. If we solve the second pair of equations, obtaining $x = 4$, $y = -1$, the result gives no clue whatever to the solution of another pair of equations of this type. Note that the solution depends only on the coefficients in the given equations.

EXERCISES

Solve the following by substituting in the above formula.

1. $6x + 8y = 15$
   $3x + 2y = 6$
2. $2x - 3y = 8$
   $4x + 5y = 20$
3. $3x - 4y = 15$
   $7x + 2y = 35$
4. $4x - 3y = 7$
   $x + 4y = 10$
5. $6x + 2y = 11$
   $3x - 4y = 3$
6. $5x - 2y = 7$
   $2x + 3y = 5$
7. $x - 5y = 9$
   $5x - 3y = 7$
8. $9x + 2y = 16$
   $2x - 7y = 2$
9. $12x - 9y = 14$
   $3x + 7y = 25$

110. Second order determinants.—There is a special scheme for writing the general solution obtained on page 100 that turns out to be of great importance. A general regularity in the letters and subscripts is apparent at once. The denominators in the values of x and y are the same. We shall study this expression first. The general scheme to which we refer is to write the denominator $a_1b_2 - a_2b_1$ as shown at the right. To make the square arrangement equal to $a_1b_2 - a_2b_1$ we agree that the terms a_1 and b_2 in the diagonal of the square running from the upper left to the lower right shall form the product a_1b_2 taken with the positive sign, while the terms a_2 and b_1 running from the lower left to the upper right shall form the product a_2b_1 taken with the negative sign. Using this scheme we have

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

which are the values of x and y obtained on page 100. Now note that the quantities in $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ may be regarded as being lifted directly out of the left members of the equations, the x and y being omitted. Note further that the square arrangement in the numerator of the value of x is exactly like the denominator with the exception that the a's (the coefficients of x) are replaced by the c's (the constant terms). In the value of y the numerator is formed in the same way by replacing the b's by the c's.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Notice that in this scheme the constant terms must be in the second members of the equations. That is, the equations must be written exactly in the form $ax + by = c$.

These square arrays of numbers are called determinants of the second order.

The great importance of this method is that a very closely analogous method can be used for solving n first degree equations in n variables. However, the method of evaluating the determinants in the higher cases is somewhat more complicated.

Example. Solve $3x - 9y = 14$, $4x + y = 7$, using determinants.

$$3x - 9y = 14 \quad (1)$$
$$4x + y = 7 \quad (2)$$

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

SOLUTION: From the determinant formula

$$x = \frac{\begin{vmatrix} 14 & -9 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -9 \\ 4 & 1 \end{vmatrix}} = \frac{14 \cdot 1 - 7(-9)}{3 \cdot 1 - 4(-9)} = \frac{14 + 63}{3 + 36} = \frac{77}{39}$$

$$y = \frac{\begin{vmatrix} 3 & 14 \\ 4 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & -9 \\ 4 & 1 \end{vmatrix}} = \frac{3 \cdot 7 - 4 \cdot 14}{3 \cdot 1 - 4(-9)} = \frac{21 - 56}{3 + 36} = -\frac{35}{39}$$

EXERCISES

Solve the following using determinants.

1. $2y - 3x = -14$
   $3y + 5x = -2$
2. $5m + 3n = 9$
   $3m - 4n = -17$
3. $2a + 7b = 18$
   $5a - 3b = 12$
4. $7x - 11y = 2$
   $4x - 10y = 0$
5. $4a - 3b = 3$
   $2a - b = 3$
6. $6x + y = 10$
   $3x + 7y = 31$
7. $x + y = 3$
   $5x - 11y = 10$
8. $7m + 2n = 12$
   $-2m + n = -7$

Before the general formula can be used the equations must be reduced to standard form. The examples below are typical.

$$5(x + 2y) - (2x + 5y) = 38 \quad (1)$$
$$3x + 4y - 5(2x - y) = -6 \quad (2)$$
$$3x + 5y = 38 \quad (3)$$
$$7x - 9y = 6 \quad (4)$$

$$\frac{3x - y}{4} + \frac{x + 4y}{3} = 5 \quad (1)$$
$$\frac{x + 2y}{2} - \frac{5x - 3y}{3} = 7 \quad (2)$$
$$13x + 13y = 60 \quad (3)$$
$$-7x + 12y = 42 \quad (4)$$

In each case equations (1) and (2) are reduced to the standard form (3) and (4). Solve each of these sets of equations. Also solve the following sets.

9. $7x - 2(c + 1) = 3x - 2$
   $3(c - 1) + 6x = 2(x + 1)$
10. $5(x - 2) + 2(y + 6) = 15$
    $3x - 5(3 - 2y) = 30$
11. $4(x - 1) + 3(y + 1) = 36$
    $3(2x - 1) - 2(3y + 4) = 96$
12. $2(3x + y) + x - y = 28$
    $4(x + y) - 5(x - y) = 20$

111. Linear equations in three unknowns.—The solution of three equations in three unknowns may be reduced to the solution of two equations in two unknowns by eliminating any one of the unknowns between any pair of the equations, and then eliminating the *same* unknown from either of the remaining two pairs of equations. For example, if x is eliminated between the first and second equations and also between the second and the third, then each of the resulting equations is linear in y and z.

The steps are shown in the following.

Example. Solve the equations at the right.

STEP 1: Use equations (1) and (2) to eliminate x, obtaining equation (4).

STEP 2: Use equations (2) and (3) to eliminate x, obtaining equation (5).

STEP 3: Solve (4) and (5) obtaining $z = 3$ and $y = -1$.

STEP 4: Substitute for y and z in any one of the three original equations and solve for x.

$$\begin{aligned} 2x - y + 3z &= 14 \quad (1)\\ x + 3y + z &= 2 \quad (2)\\ 3x + 2y + z &= 7 \quad (3)\\ -7y + z &= 10 \quad (4)\\ 7y + 2z &= -1 \quad (5) \end{aligned}$$

$$z = 3,\ y = -1,\ x = 2$$

To check, substitute $x = 2$, $y = -1$, $z = 3$, in all three equations.

EXERCISES

Solve the following sets of equations.

1. $x - y + z = 2$
$2x + 5y - 3z = 17$
$3x + y - z = 18$

2. $-2x + 3y - 2z = -16$
$x - 4y + 3z = 3$
$3x - y - z = 5$

3. $4x - 5y + 6z = -16$
$2x + y - z = 1$
$3x + 2y - 3z = 0$

4. $8x - 9y + 2z = 44$
$2x + 3y - z = 13$
$3x - 2y + 6z = 7$

5. $2a - 3b + 7c = -10$
$a + 2b - 3c = 15$
$6a - 2b + 2c = 22$

6. $m - n + p = 4$
$3m + 6n - 5p = 30$
$-2m + 2n + 4p = -8$

7. $3p - 5q + 7r = 11$
$4p + q - 2r = 15$
$-2p + 6q - r = 8$

8. $2x + 3y + 4z = 19$
$-3x + 2y + z = 24$
$4x - 3y + 12z = -21$

7. State in words a rule for solving three linear equations in three unknowns.

112. The number of unknowns and the number of equations.—We have solved two equations in two unknowns and three equations in three unknowns. These are instances of the general fact that to determine a certain number of unknowns there must be the same number of independent relations, or equations. Thus, to determine four unknowns there must be four equations, and so on.

That is, we have found that there is an infinite set of values of x and y that satisfy one equation in two unknowns. Hence it follows that there must be at least two equations in order to determine two unknowns. But we have also seen that there is only one pair of values of the unknowns that satisfy two linear equations in x and y (see page 96), and that these values may be found by the ordinary methods of solving the equations.

If we consider two linear equations in three unknowns, as in the example at the right, it is easy to see that an infinite number of values of x, y, and z can be found that will satisfy the two equations. If we solve (1) for z, obtaining $z = 12 + y - 2x$ and substitute in (2), we have

$$2x - y + z = 12 \quad (1)$$
$$x + 2y - 3z = 8 \quad (2)$$

$$x + 2y - 3(12 + y - 2x) = 8$$
or
$$7x - y = 44$$

which is satisfied by an infinite set of values of x and y. Substituting in (1) any one of the pair of values of this set, we get a corresponding value of z.

If we had another equation in x, y, z, this would be satisfied by not more than one of the values of x, y, z thus found.

113. Dependent, independent, consistent, and inconsistent equations.—If in the system of equations at the right we eliminate x, using (1) and (2), and then eliminate x again using (1) and (3), we obtain the same equation (4) in both cases. This means that this set of equations are not independent. In fact, if we add (1) and (2) and divide by (2), we obtain (3). That is, (3) is dependent on (1) and (2).

$$2x - 3y + 6z = 16 \quad (1)$$
$$6x + y - 4z = 8 \quad (2)$$
$$4x - y + z = 12 \quad (3)$$
$$5y - 11z = -20 \quad (4)$$

Two equations may be inconsistent. That is, there may be no values of the unknowns that satisfy both of them. Thus the equations at the right are inconsistent. They represent two parallel lines, which do not meet.

$$\begin{aligned} x + y &= 4 \quad (1) \\ x + y &= 6 \quad (2) \end{aligned}$$

In order that a set of equations shall have a definite solution the equations must be independent and consistent. Later we shall see what happens when we attempt to solve a set of dependent equations and also a set of inconsistent equations. In general, a set of inconsistent equations has no solution and a set of dependent equations has an infinity of solutions. The inconsistent set gives a solution of the type $a/0$, which is not a number; and a dependent set gives a solution of the type $0/0$, which is any number whatever.

It will be interesting simply to note the following. A first degree equation in x, y, z may be represented by a plane as an equation in x, y is represented by a straight line. If these planes meet in exactly one point, the equations are independent and consistent; if the planes all meet in the same straight line, the equations are dependent; and if there is no point on all three planes, two of the planes being parallel for example, then the equations are inconsistent.

EXERCISES

1. Find at least three sets of values of x, y, z that will satisfy the two equations at the right.

$$\begin{aligned} 2x - y + z &= 10 \quad (1) \\ x + 4y - z &= 12 \quad (2) \end{aligned}$$

2. Compare the set of values of x, y, z that will satisfy the two equations $x - 3y + z = 12$ and $3x - 9y + 3z = 36$.

3. In the set of equations at the right, (3) is obtained by adding the members in (1) and (2). If we have any set of values of x, y, z that satisfy (1) and (2), will this set of values satisfy (3)? How many sets of values can be found that will satisfy these three equations? This is a simple case of dependent equations.

$$\begin{aligned} 3x - 2y + 5z &= 18 \quad (1) \\ 2x + 4y - 6z &= 15 \quad (2) \\ \hline 5x + 2y - z &= 33 \quad (3) \end{aligned}$$

4. Suppose we use equation (4) in which the 33 in (3) is replaced by 40. Is it possible to find a set of values of x, y, z that will satisfy (1), (2), (4)? Equations (1), (2), (4) are a simple case of inconsistent equations.

$$5x + 2y - z = 40 \quad (4)$$

114. General solution of three linear equations.—The work of solving such a set of equations is a little complicated and we shall give only the results; below, the same results are given in the form of determinants.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \quad (1)\\ a_2x + b_2y + c_2z &= d_2 \quad (2)\\ a_3x + b_3y + c_3z &= d_3 \quad (3)\end{aligned}$$

In the set of equations at the right one of the unknowns, such as z, may be eliminated from (1) and (2) and also from (2) and (3), thus giving two equations in x and y. These may then be solved by the usual method of solving two linear equations. The final results are:

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

We now note the following.

1. The three denominators are exactly the same.

2. The numerator in the value of x is the same as the denominator except that the a's (the coefficients of x in the equations) are replaced by the d's. In the values of y and z the b's and the c's of the denominators are replaced in turn by the d's.

115. Third order determinants.—The solution of the three equations at the right may be arranged in a manner similar to that given on page 101 for the solution of two equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1\\ a_2x + b_2y + c_2z &= d_2\\ a_3x + b_3y + c_3z &= d_3\end{aligned}$$

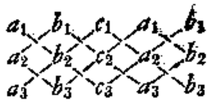
The solution is written thus:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}$$

with the understanding that these square arrays are to be interpreted according to the following scheme. Consider the square array

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Repeat the first two columns, forming the five columns shown at the right. Let the three numbers connected by the lines running from the upper left to the lower right form the three



products $a_1b_2c_3$, $b_1c_2a_3$, $c_1a_2b_3$, with the plus sign; and the three numbers connected by the remaining three dotted lines form the three products $a_3b_2c_1$, $b_3c_2a_1$, $c_3a_2b_1$, with the negative signs.

Then the algebraic sum of these six products is to be the value of the given square array. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

This square array, so interpreted, is a determinant of the third order. The six terms are called the expansion or evaluation of the determinant. When the solutions written in determinant form above are "expanded" according to this scheme, the results are exactly those given on page 106.

Note that the same determinant is used for all these denominators. It is formed by taking the coefficients of the unknowns exactly in the relation in which they stand in the equations. Note also that the numerators are formed by replacing in the common denominator the a's, b's, and c's in succession by the d's of the equations.

Note further that the a's are the coefficients of x in the three equations, the b's are the coefficients of y, the c's are the coefficients of z. The subscript $_1$ refers to the first equation, the subscript $_2$ refers to the second equation, and the subscript $_3$ refers to the third equation. Compare page 101. This is a standard notation that can easily be extended to larger numbers of equations.

To use the above formulas, the equations must be reduced to the standard form $ax + by + cz = d$. Compare page 101.

Example. Solve the equations at the right, using determinants.

$$\begin{aligned} 6x - 3y + z &= 23 \\ 2x + 5y - 3z &= -5 \\ 3x + 2y + 4z &= 15 \end{aligned}$$

SOLUTION:

$$x = \frac{\begin{vmatrix} 23 & -3 & 1 \\ -5 & 5 & -3 \\ 15 & 2 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & -3 & 1 \\ 2 & 5 & -3 \\ 3 & 2 & 4 \end{vmatrix}} = \frac{23 \cdot 5 \cdot 4 + (-3)(-3)15 + 1(-5)2 - 15 \cdot 5 \cdot 1 - (-5)(-3)4 - 2(-3)23}{6 \cdot 5 \cdot 4 + (-3)(-3)3 + 1 \cdot 2 \cdot 2 - 3 \cdot 5 \cdot 1 - 2(-3)6 - 2(-3)4} = \frac{588}{196} = 3$$

$$y = \frac{\begin{vmatrix} 6 & 23 & 1 \\ 2 & -5 & -3 \\ 3 & 15 & 4 \end{vmatrix}}{196} = \frac{6(-5)4 + 23(-3)3 + 1 \cdot 2 \cdot 15 - 3(-5)1 - 15(-3)6 - 4 \cdot 23 \cdot 2}{196} = \frac{-196}{196} = -1$$

$$z = \frac{\begin{vmatrix} 6 & -3 & 23 \\ 2 & 5 & -5 \\ 3 & 2 & 15 \end{vmatrix}}{196} = \frac{6 \cdot 5 \cdot 15 + (-3)(-5)3 + 23 \cdot 2 \cdot 2 - 3 \cdot 5 \cdot 23 - 2(-5)6 - 15(-3)2}{196} = \frac{392}{196} = 2$$

Note that the denominators need to be computed only once since they are exactly the same.

In Chapter 25 we shall study determinants further and shall find short methods for evaluating them.

We shall state here, however, that the solution of n linear equations in n unknowns may be written down directly as quotients of nth order determinants. The evaluation of such determinants is by no means so direct as the scheme used above for second and third order determinants.

EXERCISES

Evaluate determinants.

1. $\begin{vmatrix} 4 & -8 \\ 12 & 16 \end{vmatrix}$
2. $\begin{vmatrix} 2 & 3 \\ 3 & -4 \end{vmatrix}$
3. $\begin{vmatrix} 1 & -5 \\ 11 & 12 \end{vmatrix}$
4. $\begin{vmatrix} 6 & -4 \\ -3 & 2 \end{vmatrix}$
5. $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -7 & 4 \\ -2 & 1 & -3 \end{vmatrix}$
6. $\begin{vmatrix} -2 & 0 & 2 \\ -1 & 3 & 1 \\ 4 & -2 & 3 \end{vmatrix}$
7. $\begin{vmatrix} a & b & 3 \\ c & a & 4 \\ b & c & 6 \end{vmatrix}$

8. $\begin{vmatrix} 5 & 3 & -1 \\ 1 & -2 & 0 \\ 4 & -6 & 2 \end{vmatrix}$ 9. $\begin{vmatrix} -1 & -1 & 6 \\ -2 & 5 & 2 \\ 4 & -1 & 17 \end{vmatrix}$ 10. $\begin{vmatrix} x & 3 & x \\ 5 & x & 6 \\ -2 & x & 7 \end{vmatrix}$

11. $\begin{vmatrix} 8 & 0 & 9 \\ 4 & -1 & 6 \\ 3 & 2 & -5 \end{vmatrix}$ 12. $\begin{vmatrix} 5 & 3 & 1 \\ 2 & 0 & -2 \\ -1 & 3 & 4 \end{vmatrix}$ 13. $\begin{vmatrix} x^2 & x & 1 \\ a & b & c \\ d & e & a \end{vmatrix}$

Solve the following equations, using determinants.

14. $2x = -3y + 7$
$2x + y = 1$

15. $3x - 4y = 14$
$4y = 2 - 5x$

16. $7y - 3x - 6 = 0$
$6x + 4y = 6$

17. $8m - 7 = -9z$
$4m - 2z + 3 = 0$

18. $16m + 3n = 11$
$20m = 30 - 7n$

19. $\frac{3}{2}x - \frac{4}{3}y + 6 = 0$
$\frac{4}{5}x + \frac{2}{3}y = -9$

20. $x + y - 2z = 7$
$2x - 3y - 2z = 0$
$x - 2y - 3z = 3$

21. $6s + 3t + 2w = 1$
$3s - 4w = 4$
$5s - t = 14$

22. $2m + 3n + 2p = 2$
$4m - n - p = 1$
$2m + n + 2p = 4$

23. $7a - 3b + 2c = -45$
$2a + 8b - 5c = 37$
$-3a + 4b + 6c = 26$

24. $-x - 3y + 5z = 12$
$4x + 7y + 8z = 8$
$3x + 2y - 7z = -20$

25. $5z - 9y = 16$
$7y + 3x = 18$
$15x - 8y = 12$

26. $3x + y = 10$
$15y + 9z = 6$
$9x + 4z = 23$

27. $5x - 23y - 8z = -7$
$3x + 6y + 4z = 9$
$7x + 6y + 13z = 20$

28. $2(x - 1) - 3(y + 1) + 5(z - 6) = 20$
$\frac{x+1}{2} + \frac{y-1}{3} - \frac{z-1}{4} = 1$
$3(x + y - z) - 3(x - y + z) + 4(x - y - z) = 16$

29. $5(x - y) - 2(y - z) + 3(z - x) = 15$
$4x - 2(x - 2y + z) + 3(-x - y + 5z) = 30$
$\frac{x}{4} + \frac{x+y}{3} + \frac{x+y+z}{2} = 10$

PROBLEMS

The two groups of problems, A and B, are of about equal difficulty. Either group may be used.

Group A

1. An airplane flies 180 miles in $1\frac{1}{2}$ hours going with the wind and returns in 3 hours going against a wind with the same velocity; find the rate of the wind and the rate of the airplane in still weather.

2. A milkman has one kind of milk containing 3.2% of butter fat and another containing 4.8%; how many pounds of each kind must he use for 1000 pounds which will contain 4.2% of fat?

3. If a day's ration should contain 4.5 ounces protein, 2 ounces fat, and 18 ounces carbohydrates, and if it is made of whole-wheat cereal, milk, and eggs, how many ounces of each are needed, it being given that cereal contains 14% protein, 2% fat, 72% carbohydrate; milk, 3% protein, 4% fat, 5% carbohydrate; eggs, 15% protein, 10% fat?

4. A cyclist, A, rides a distance of 28 miles in the time another, B, rides 18 miles. C's rate of riding is 3 miles per hour less than A's and 2 miles greater than B's; what is C's rate?

5. When we divide a certain two-digit number by its tens' digit, the result is 13. If we reverse digits in the number and then divide the result by the original number plus 2, the quotient is 31/14. Find the original number.

6. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

7. The same number is added to each of the numbers 8, 9, 10, 12. What is the number if the product of the first and last sums is equal to the product of the second and third sums?

8. A steamer going with the tide makes 19 miles per hour, and going against a current half as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

9. A beam is 12 feet long. It carries a 40-pound weight at one end, a 60-pound weight 3 feet from this end, and a 70-pound weight at the other end. Where is the fulcrum if the beam is balanced?

10. A bicyclist starts out riding 12 miles per hour, and is followed 40 minutes later by another riding 16 miles per hour. Find when they will be 5 miles apart. (Two answers.)

11. One-third of the sum of two numbers is 14, and one-half of their difference is 4; find the numbers.

12. Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator, and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from the denominator.

13. In a bag containing black and white balls, half the number of white balls is equal to a third of the number of the black balls; and twice the whole number of balls exceeds three times the number of black balls by 4. How many balls does the bag contain?

14. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits are interchanged, the number is diminished by 180; if the left-hand digit is halved and the middle and right-hand digits are interchanged, the number is diminished by 454; find the number.

15. Two persons, 27 miles apart, setting out at the same time are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions; find their rates of walking.

16. The sum of the angles of a triangle is 180°. If A, B, C are the angles, and if it is given that $2A - 2B + 3C = 100°$, $A + 5B - 2C = 360°$, find the three angles.

In setting up equations, symbols indicating units of measure are usually omitted. At the end of the solution we know that the values are, in this case, degrees, and we label the answers accordingly.

17. If a three-digit number is divided by the sum of its digits, the quotient is 26. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the digits in ones' and tens' places are interchanged, the number is increased by 18. Find the number.

18. Two persons start at the same time from two stations a miles apart. If they walk in the same direction they will be together in b hours and if they walk in opposite directions they will be together in c hours. Find their rates of walking. Use the formula for solving problem 15.

19. The earth and Jupiter are in conjunction on a certain date. Find in years the length of time before they will be in conjunction again, the period of Jupiter being 12 years.

Group B

1. A boatman rowing down a river makes 23 miles in 3 hours and returns at the rate of $3\frac{1}{2}$ miles per hour. How fast does the river flow?

2. The earth and Mars were in conjunction July 12, 1907. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days?

3. In a bicycle race A starts 32 minutes ahead of B. B rides at the rate of $20\frac{1}{3}$ miles per hour, while A rides $18\frac{2}{5}$ miles per hour. How many miles from the starting point does B overtake A?

4. There is a rectangle whose length is 60 feet more, and whose width is 20 feet less, than the side of a square of equal area. Find the dimensions of the square and the rectangle.

5. The middle digit of a number between 100 and 1000 is zero, and the sum of the other digits is 11. If the digits are reversed, the number so formed exceeds the original number by 495. Find the original number.

6. A milkman sells milk containing 3.6% butter fat and another kind containing 4.5%. If he has 500 gallons of milk containing 4.1% butter fat, how many gallons of each kind can he get from this milk by extracting butter fat from part of it and adding to the other?

7. Half the sum of two numbers is 20, and three times their difference is 30; find the numbers.

8. Four times B's age exceeds A's age by twenty years, and one-third of A's age is less than B's age by two years; find their ages.

9. If 1 is added to the numerator of a fraction, it reduces to a fraction equal to 1/5; if 1 is taken from the denominator, it reduces to 1/7. Required the fraction.

10. A certain number of two digits is three times the sum of its digits, and if 45 is added to it the digits will be reversed; find the number.

11. A grocer wishes to mix sugar at 8 cents a pound with another sort at 5 cents a pound to make 60 pounds to be sold at 6 cents a pound. What quantity of each must he take?

12. A takes 3 hours longer than B to walk 30 miles; but if he doubles his pace he takes 2 hours less time than B; find their rates of walking.

13. If the numerator of a fraction is increased by 2 and the denominator by 1, it equals $\frac{5}{8}$, and if the numerator and denominator are each diminished by 1, it equals $\frac{1}{2}$; find the fraction.

14. In an athletic meet, team A got 33 points by getting 5 firsts, 2 seconds, and 2 thirds; team B got 25 points by getting 3 firsts, 3 seconds, and 1 third; and team C got 23 points by getting 1 first, 4 seconds, and 6 thirds. How many points did each place count?

15. The relation between the temperature readings of the centigrade and the Fahrenheit thermometers is given by the equation $F = 9/5\ (C + 32)$, where C is the reading on the centigrade and F the reading of the same temperature on the Fahrenheit. Solve the equation for C.

16. The same number is added to each of the numbers a, b, c, d. What is the number if the product of the first two sums is equal to the product of the last two sums?

17. If in problem 16 the sum of the squares of the first two sums is equal to the sum of the squares of the last two sums, what is the number added to a, b, c, d?

18. A and B can do a piece of work in 12 days, B and C can do it in 14 days, and C and A can do it in 16 days. How long will it take each of them to do it?

19. If in problem 18 you use a, b, and c instead of 12, 14, 16 respectively solve the problem. Then find the answer for problem 18 by substituting in the formula that you obtain.

CHAPTER 9:

RADICALS; FRACTIONAL AND NEGATIVE EXPONENTS

So far we have used only positive integers as exponents. In practice, however, fractions and negative numbers are used freely as exponents, and it is therefore necessary to study these in detail. We shall find that the radical sign may be replaced entirely by such exponents, but that it is highly convenient to use both. We shall start this chapter with a further study of radicals.

116. The principal root of a number.—Every positive number has two square roots, one positive and one negative. Thus 2 and -2 are square roots of 4; and a and $-a$ are square roots of a^2. The square root sign over a positive number indicates its positive square root. Thus, while 2 and -2 are both square roots of 4, it is not correct to write $\sqrt{4} = \pm 2$. The negative square root of 4 may be indicated by $-\sqrt{4}$.

$$\sqrt{4} = 2$$
$$\sqrt{a^2} = a$$
$$\sqrt{(-a)^2} = a$$
$$\sqrt[3]{8} = 2$$
$$\sqrt[4]{16} = 2$$

Also 8 has one positive cube root, 16 has one positive fourth root, and in general any positive number has one positive nth root.

Again, a negative number has one real odd root, which is negative. Thus $\sqrt[3]{-8} = -2$, $\sqrt[5]{-32} = -2$, $\sqrt[3]{-27} = -3$.

Let a be a positive number. Then $\sqrt[n]{a}$ will be used to represent the positive nth root of a, and in case n is odd, $\sqrt[n]{-a}$ will be used to represent the negative nth root of $-a$. These are called the principal nth roots of a and $-a$ respectively. We also speak of the principal root as "the" root of a number. In case n is even we do not speak of the "principal" root in $\sqrt[n]{-a}$. Such roots will be considered later. See pages 130, 131.

An expression of the form $\sqrt[n]{a}$, a and n being any numbers, is called a radical; a is called the radicand, and n the index of the radical.

In the $\sqrt[n]{a^m}$, which is the nth root of a^m, a is first raised to the mth power and then the principal nth root is taken. In case a is negative, the order of these operations is not always reversible. To avoid confusion in the discussion that follows, a will be positive unless otherwise stated.

$$\sqrt[n]{x} = \sqrt[n]{y} \quad \therefore x = y$$

It follows by simple multiplication that if $\sqrt[n]{x} = \sqrt[n]{y}$, then $x = y$.

117. ***Laws of exponents.***—When m and n are positive integers we can easily prove the laws of exponents given at the right. In II it is assumed for the present that m is greater than n.

$$\begin{array}{rcll} a^m \cdot a^n &=& a^{m+n} & \text{I} \\ a^m \div a^n &=& a^{m-n} & \text{II} \\ (a^m)^n &=& a^{mn} & \text{III} \\ (ab)^m &=& a^m \cdot b^m & \text{IV} \\ \left(\frac{a}{b}\right)^m &=& \frac{a^m}{b^m} & \text{V} \end{array}$$

Laws I and II have already been proved. See page 34, §45.

By definition $(a^m)^n = a^m \cdot a^m \ldots$ (to n factors a^m) $= a \cdot a \cdot \ldots$ (to mn factors a), and hence $(a^m)^n = a^{mn}$, which proves III.

To prove IV, we have

$(ab)^m = ab \cdot ab \cdot \ldots$ (to m factors ab) $= (a \cdot a \cdot \ldots$ [to m factors a])($b \cdot b \cdot \ldots$ [to m factors b]). Hence $(ab)^m = a^m b^m$. In this proof we use the commutative and associative laws of factors.

Finally, using the rule for multiplication of fractions, we have

$$\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \ldots \left(\text{to } m \text{ factors } \frac{a}{b}\right) = \frac{a^m}{b^m}.$$

EXERCISE

1. State in words the five laws of exponents given above. Thus IV is: The mth power of a product is the product of the mth power of the factors.
2. Give the authority for each step in the proofs suggested above. Which ones of the equations on page 17 do you use?

118. ***Definition of positive fractional exponents.***—Clearly the meaning of a positive integral exponent as showing the number of times the base is used as a factor cannot apply unless the exponent is a positive integer. We cannot use a as a factor $\frac{3}{4}$ times or -4 times. Hence we must make new definitions of the meaning of such exponents. These definitions must be such as to make the laws of exponents, I, II, . . . , V, apply to these new exponents. We shall obtain our definitions by assuming that law I holds.

Let p/q be a fraction whose terms are both positive integers. If we add exponents in multiplying $a^{1/2} \cdot a^{1/2}$, we see that $a^{1/2}$ is one of the two equal factors of a, and hence $a^{1/2} = \sqrt{a}$. Similarly, $a^{1/3} = \sqrt[3]{a}$, and $a^{2/3} = \sqrt[3]{a^2}$. In this way we can show that $a^{p/q} \cdot a^{p/q} \cdot \ldots$ (to q factors $a^{p/q}$) $= a^p$ and that hence $a^{p/q}$ is one of the q equal factors of a^p, and therefore $a^{p/q} = \sqrt[q]{a^p}$. That is, $a^{p/q}$ is the qth root of a^p.

$$\begin{aligned} a^m \cdot a^n &= a^{m+n} \qquad \text{I} \\ a^{1/2} \cdot a^{1/2} &= a^1 = a \\ \therefore a^{1/2} &= \sqrt{a} \\ a^{1/3} \cdot a^{1/3} \cdot a^{1/3} &= a^1 = a \\ \therefore a^{1/3} &= \sqrt[3]{a} \\ a^{2/3} \cdot a^{2/3} \cdot a^{2/3} &= a^2 \\ \therefore a^{2/3} &= \sqrt[3]{a^2} \\ a^{p/q} &= \sqrt[q]{a^p} \end{aligned}$$

To work freely with fractional exponents we must at times reduce them to higher or lower terms. Thus we must use $a^{1/2} = a^{2/4} = a^{3/6}$, and in general $a^{p/q} = a^{mp/mq}$. At this point this will be assumed without proof.

When a is negative this does not hold in general. Thus $(-8)^{1/3} = -2$, while $(-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$. In this text, $a^{p/q}$, or $\sqrt[q]{a^p}$, will mean that a is raised to the pth power and then the qth root taken.

119. Definition of negative and zero exponents.—If k is any positive rational number, and if $a^{2k} \cdot a^{-k} = a^k$, then $a^{-k} = \dfrac{a^k}{a^{2k}} = \dfrac{1}{a^k}$. It follows that for any such k, a^{-k} may be replaced by $\dfrac{1}{a^k}$, and $\dfrac{1}{a^{-k}}$ by a^k.

$$\begin{aligned} a^{2k} \cdot a^{-k} &= a^k \\ \therefore a^{-k} &= \frac{1}{a^k} \end{aligned}$$

If $a^k \cdot a^0 = a^k$, then $a^0 = 1$. Hence any number, other than 0, with an exponent zero is equal to unity. These two definitions follow directly from the requirements that law I shall hold.

$$\begin{aligned} a^k \cdot a^0 &= a^k \\ a^0 &= 1 \end{aligned}$$

A formal proof that laws II, . . . V (page 114) hold for fractional and negative exponents will not be given here. That they do hold is easily verified for particular cases. That $a^{m/n} = a^{km/kn}$ may also be verified. Thus $a^{1/2} = (a^3)^{1/6}$ and $a^{1/3} = (a^2)^{1/6}$.

120. Powers and roots of monomials.—In the following when roots are required give the principal roots.

EXERCISES

Evaluate:

1. 3^2 2. $\left(\frac{3}{4}\right)^2$ 3. 7^2 4. $\left(\frac{2}{5}\right)^2$ 5. $1 \cdot 1^2$

6. $(-5)^2$ 7. $(-5)^3$ 8. $\left(\frac{1}{2}\right)^5$ 9. $\left(\frac{1}{2}\right)^6$ 10. $\left(-\frac{1}{2}\right)^6$

11. $4^{1/2}$ 12. $\left(\frac{4}{9}\right)^{1/2}$ 13. $(1.21)^{1/2}$ 14. $(-8)^{1/3}$ 15. $(-8)^{2/3}$

16. $\left(\frac{16}{625}\right)^{1/4}$ 17. $\left(\frac{16}{625}\right)^{3/4}$ 18. $\left(\frac{8}{-27}\right)^{2/3}$ 19. $(-8)^{1/3} \cdot (16)^{1/2}$

20. $16^{3/4} \cdot 8^{2/3}$

Supply the missing words in the following.

21. To find the square root of a product take the ______ of the ______ ______ of the factors.

22. To find the square root of a fraction take the quotient of the square roots of ______ ______ of the fraction.

23. To find the nth root of a product take the ______ of the nth roots of the ______ of the product.

24. If a is positive and n is a positive integer, then $\sqrt[n]{a}$ has a real positive root for ______ values of n.

25. If a is negative, then $\sqrt[n]{a}$ has a real value for ______ values of n. Is this value positive or negative? Under the same condition, $\sqrt[n]{a}$ has no real values for ______ values of n.

121. Distributing an exponent over the factors of a product.—In using $(ab)^k = a^k b^k$, we may think of distributing the exponent k over the factors a and b. Thus in $(a^2b^3c^4)^{1/8} = a^{1/4}b^{3/8}c^{1/2}$ we distribute the exponent 1/8 over the factors a^2, b^3, c^4.

Note that an exponent may be distributed over the factors of a product, but not over the addends of a sum. Thus $(ab)^{1/2} = a^{1/2}b^{1/2}$, but *not* $(a + b)^{1/2} = a^{1/2} + b^{1/2}$. Using this idea, we have for instance $(16 \times 49)^{1/2} = 4 \cdot 7 = 28$ and $\left(\frac{64}{81} \times \frac{144}{625}\right)^{1/2} = \frac{8}{\underset{3}{\cancel{9}}} \cdot \frac{\overset{4}{\cancel{12}}}{25} = \frac{32}{75}$.

122. Surds; simplifying surds with integral radicands.—An indicated root that cannot be expressed exactly without a square root sign or a fractional exponent, or equivalent symbols or words, is called a surd. Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt{1/2}$, $8^{1/2}$, $16^{2/3}$ are surds, while $\sqrt{4} = 2$, $\sqrt[3]{8} = 2$, $\sqrt[3]{27/64} = \frac{3}{4}$ are not surds. A surd that is an indicated square root is called a quadratic surd, and a surd that is an indicated cube root is called a cubic surd. Under certain circumstances a surd with an integral radicand may be simplified as in the following.

Example 1. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$

Example 2. $\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$

Example 3. $\sqrt[n]{a^n b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}$

$$\boxed{\sqrt[n]{a^n b} = a\sqrt[n]{b}}$$

To simplify a quadratic surd, if possible, factor the expression under the radical into two factors one of which is a square. It will then be in the form $\sqrt{a^2 b}$. The factor b should contain no square factor. Then the simplified surd is $a\sqrt{b}$. The factor a^2 must of course not be unity, since that will effect no simplification.

EXERCISES

Simplify the following surds.

1. $\sqrt{12}$	2. $\sqrt{18}$	3. $\sqrt{24}$	4. $\sqrt{32}$	5. $\sqrt{20}$
6. $\sqrt{28}$	7. $\sqrt{40}$	8. $\sqrt{44}$	9. $\sqrt{48}$	10. $\sqrt{60}$
11. $\sqrt{27}$	12. $\sqrt{45}$	13. $\sqrt{54}$	14. $\sqrt{63}$	15. $\sqrt{72}$
16. $\sqrt{80}$	17. $\sqrt{96}$	18. $\sqrt{90}$	19. $\sqrt{200}$	20. $\sqrt{162}$
21. $\sqrt{176}$	22. $\sqrt{117}$	23. $\sqrt{160}$	24. $\sqrt{125}$	25. $\sqrt[3]{24}$
26. $\sqrt[3]{40}$	27. $\sqrt[3]{48}$	28. $\sqrt[3]{56}$	29. $\sqrt[3]{72}$	30. $\sqrt[3]{80}$
31. $\sqrt[4]{32}$	32. $\sqrt[4]{48}$	33. $\sqrt[4]{64}$	34. $\sqrt[4]{80}$	35. $\sqrt[4]{96}$
36. $\sqrt[5]{64}$	37. $\sqrt[5]{96}$	38. $\sqrt[5]{128}$	39. $\sqrt[5]{192}$	40. $\sqrt[5]{288}$

41. $\sqrt[3]{a^4b^2}$	42. $\sqrt{ab^4c^3}$	43. $\sqrt[4]{m^2b^3c^4}$	44. $\sqrt[4]{p^6q^5r^8s^4}$
45. $\sqrt[3]{k^4l^6m^8}$	46. $\sqrt[3]{a^9b^8c^7}$	47. $\sqrt[4]{r^6p^9q^{10}}$	48. $\sqrt[5]{m^5n^6p^{10}}$

49. $\sqrt{(a-4)^2(a+2)}$	50. $\sqrt{(x^2-y^2)(x+y)}$
51. $\sqrt[3]{(a^2-b^2)(a+b)^2}$	52. $\sqrt{a^3(x^2-2xy+y^2)}$
53. $\sqrt{(x+4)(x^2+7x+12)}$	54. $\sqrt{(x-1)(x^2-1)}$
55. $\sqrt[3]{(x^2-5x+4)(x-1)^2}$	56. $\sqrt{(x^4+4x^3+4x^2)x}$
57. $\sqrt[3]{x^2(x^3+7x^2-8x)}$	58. $\sqrt{(a^3-b^3)(a^2-b^2)}$
59. $\sqrt{(x^3+y^3)(x^2-y^2)}$	60. $\sqrt[3]{27x^4y^5z^6(x-y)^4}$

123. Simplifying surds with fractional radicands.—A surd with a fractional radicand may be simplified as shown in the following.

Example 1. $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4}} \cdot \sqrt{2} = \frac{1}{2}\sqrt{2}$

Example 2. $\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{18}{27}} = \sqrt[3]{\frac{1}{27}} \cdot \sqrt[3]{18} = \frac{1}{3}\sqrt[3]{18}$

Example 3. $\sqrt[4]{\frac{3}{8}} = \sqrt[4]{\frac{6}{16}} = \sqrt[4]{\frac{1}{16}} \cdot \sqrt[4]{6} = \frac{1}{2}\sqrt[4]{6}$

Example 4. $\frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7} \cdot \sqrt{3}}{3} = \frac{\sqrt{21}}{3}$

$$\sqrt{\frac{p}{q}} = \frac{1}{q}\sqrt{pq}$$
$$\sqrt[3]{\frac{p}{q}} = \frac{1}{q}\sqrt[3]{pq^2}$$
$$\sqrt[n]{\frac{p}{q}} = \frac{1}{q}\sqrt[n]{pq^{n-1}}$$

The purpose of this simplification is to obtain an integral expression under the radical sign. In Example 1, both terms of the fraction are multiplied by 2, which makes the denominator a perfect square. In Example 2, each term is multiplied by 9, which makes the denominator a cube. In Example 3, each term is multiplied by 2, which makes the denominator a fourth power. In Example 4, both terms are multiplied by $\sqrt{3}$. This fraction may also be written $\sqrt{\frac{7}{3}}$ and then simplified as in Example 1.

This process is called rationalizing the denominator.

In general, if the index of the root is n, ($\sqrt[n]{\ }$), then the terms of the fraction are multiplied by the smallest number that makes the denominator an nth power.

Rationalizing the denominator of a fraction may simplify its evaluation very considerably. Suppose we are required to find the value of $\frac{\sqrt{7}}{\sqrt{3}}$ correct to four decimal places. The direct method is to find $\sqrt{7} = 2.64575$ and $\sqrt{3} = 1.73205$ and then divide, three fairly tedious operations. But if this fraction is changed to $\frac{\sqrt{21}}{3}$ (see Example 4), all we need to do is to find $\sqrt{21} = 4.5826$ and then divide by 3.

EXERCISES

Rationalize denominators in the following.

1. $\sqrt{\frac{2}{3}}$ 2. $\sqrt{\frac{3}{5}}$ 3. $\sqrt{\frac{4}{3}}$ 4. $\sqrt{\frac{4}{5}}$ 5. $\sqrt{\frac{1}{6}}$ 6. $\sqrt{\frac{5}{6}}$

7. $\sqrt{\frac{3}{8}}$ 8. $\sqrt{\frac{5}{12}}$ 9. $\sqrt{\frac{4}{27}}$ 10. $\sqrt{\frac{8}{27}}$ 11. $\sqrt{\frac{7}{32}}$ 12. $\sqrt{\frac{8}{75}}$

13. $\frac{\sqrt{8}}{\sqrt{125}}$ 14. $\frac{\sqrt{7}}{\sqrt{8}}$ 15. $\frac{\sqrt{20}}{\sqrt{27}}$ 16. $\frac{\sqrt{45}}{\sqrt{128}}$ 17. $\frac{\sqrt{72}}{\sqrt{125}}$ 18. $\frac{\sqrt{98}}{\sqrt{243}}$

19. Given $\sqrt{3} = 1.73205$, $\sqrt{6} = 2.44949$, $\sqrt{15} = 3.87298$, $\sqrt{26} = 5.09902$, $\sqrt{42} = 6.48074$, $\sqrt{78} = 8.83176$, find correct to four decimals:

$$\sqrt{\frac{2}{3}},\ \sqrt{\frac{3}{5}},\ \frac{\sqrt{5}}{\sqrt{3}},\ \frac{\sqrt{7}}{\sqrt{6}},\ \frac{\sqrt{6}}{\sqrt{13}},\ \frac{\sqrt{8}}{\sqrt{13}},\ \frac{\sqrt{6}}{\sqrt{8}},\ \frac{\sqrt{18}}{\sqrt{32}}$$

Rationalize denominators:

20. $\sqrt{\frac{a}{b^3}}$ 21. $\sqrt{\frac{x^3}{y^3}}$ 22. $\sqrt{\frac{a}{(a+b)^3}}$ 23. $\sqrt{\frac{abc}{x^3y^4z^5}}$

24. $\sqrt[3]{\frac{a}{b^2}}$ 25. $\sqrt[3]{\frac{a+b}{(a-b)^2}}$ 26. $\sqrt[3]{\frac{x^2}{(x+y)^4}}$ 27. $\sqrt[3]{\frac{ab^2c^3}{x^3y^4z^5}}$

28. $\sqrt[3]{\frac{a^2-3a+2}{(a-2)(a-1)^2}}$ 29. $\sqrt[3]{\frac{a}{(a-b)^2(a^2-b^2)}}$ 30. $\sqrt[3]{\frac{(x+y)^4}{(x-y)^4}}$

31. $\sqrt{\frac{r}{(p^2-q^2)(p+q)^3}}$ 32. $\sqrt{\frac{(a-b)(a+b)}{c^3-2c^2+c}}$ 33. $\sqrt{\frac{x^4-7x^3+8x^2}{(x^2-1)(x-1)^3}}$

34. $\frac{l-m}{\sqrt{l^3-3l^2m+3lm^2-m^3}}$ 35. $\frac{\sqrt{p+q}}{\sqrt{(p+q)^3(p^2-q^2)(p-q)}}$

124. Introducing a factor under the radical sign.—In some cases it is convenient to proceed as in the following examples.

Example 1. $2\sqrt{2} = \sqrt{2^2} \cdot \sqrt{2} = \sqrt{4} \cdot \sqrt{2} = \sqrt{8}$
Example 2. $2\sqrt[3]{2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = \sqrt[3]{16}$
Example 3. $a\sqrt[n]{b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = \sqrt[n]{a^nb}$

$$\boxed{a\sqrt[n]{b} = \sqrt[n]{a^nb}}$$

EXERCISES

In each of the following, introduce under the radical the factor given before the radical.

1. $3\sqrt{2}$ 2. $2\sqrt{3}$ 3. $7\sqrt{2}$ 4. $2\sqrt{7}$ 5. $a\sqrt{2}$
6. $a\sqrt{b}$ 7. $2\sqrt[3]{2}$ 8. $2\sqrt[4]{2}$ 9. $3\sqrt[3]{3}$ 10. $2\sqrt[5]{10}$
11. $ab\sqrt{c}$ 12. $2a\sqrt{2}$ 13. $3b\sqrt[3]{2}$ 14. $5a\sqrt[3]{2b^2}$
15. $\frac{1}{2}\sqrt{2}$ 16. $\frac{1}{2}\sqrt[3]{2}$ 17. $\frac{2}{3}\sqrt{3}$ 18. $\frac{3}{4}\sqrt{18}$
19. $(a+b)\sqrt{a+b}$ 20. $(a+b)\sqrt{a-b}$ 21. $(a-b)\sqrt{a-b}$

125. Summary of simplifying radicals.—It is not possible to make a general definition of the "simplest" form of a radical that will be appropriate in all cases. A form that is "simple" for one purpose may be less "simple," or not so well adapted, for another purpose. The steps that are used in changing the form of a radical monomial have all been studied or suggested in what precedes, and a brief résumé will be made here by grouping a set of representative examples.

Example 1. $\sqrt[n]{a^n b} = a\sqrt[n]{b}$. This example is usually regarded as a simplification, but it is easy to give examples in which $\sqrt[n]{a^n b}$ will be used instead of $a\sqrt[n]{b}$. Thus $\sqrt{27}$ is often used instead of $3\sqrt{3}$.

Example 2. $\sqrt{\dfrac{a}{b}} = \dfrac{1}{b}\sqrt{ab}$. But this "simplification" is not always used. On page 162 of this book we have used, quite appropriately, $\sqrt{\dfrac{-1 - 2a \pm \sqrt{1 + 4a + 4r^2}}{2}}$ in a final answer instead of the "simplified" form $\frac{1}{2}\sqrt{-2 - 4a \pm 2\sqrt{1 + 4a + 4r^2}}$.

Example 3. $2^{1/2} \cdot 3^{1/3} = 2^{3/6} \cdot 3^{2/6} = (2^3 \cdot 3^2)^{1/6} = \sqrt[6]{72}$. For the purpose of computing approximate values this type of transformation saves labor, but for certain other purposes it is not desirable.

Example 4. $\sqrt{\dfrac{a^2b}{ac}} = \sqrt{\dfrac{ab}{c}}$. "A fraction under the radical sign should nearly always be reduced to its lowest terms." However, it may become important to consider the expression $\sqrt{\dfrac{(x-1)x}{x^2-1}}$ instead of $\sqrt{\dfrac{x}{x+1}} = \dfrac{1}{x+1}\sqrt{x(x+1)}$. If we should wish to get a decimal approximation to $\sqrt{\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{3} + \dfrac{\sqrt{4}}{4} + \dfrac{\sqrt{5}}{5} + \dfrac{\sqrt{6}}{6}}$, getting rid of the fractions under the radical sign would cause much more labor than computing the expression as it stands. The point we are making is that either member in each of the four equations given in these Examples may be the form that we wish to use. However, the second members of these equalities are usually regarded as the simplified form, and this is what we shall mean when we refer to the simplest forms of a radical. The important point for us is that we shall know how to transform radicals into the form that is best adapted to our immediate purpose.

126. Addition and subtraction of radicals.—The possibility of collecting terms containing radicals depends upon so reducing them that expressions under the radical with the same index will be the same.

$$a\sqrt{k} \pm b\sqrt{k} = (a \pm b)\sqrt{k}$$
$$a\sqrt[n]{k} \pm b\sqrt[n]{k} = (a \pm b)\sqrt[n]{k}$$

Thus $\sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$,
and $\sqrt[3]{3} + \sqrt[3]{24} = \sqrt[3]{3} + 2\sqrt[3]{3} = 3\sqrt[3]{3}$,
$\sqrt{2} + \sqrt{8} - \sqrt{32} = \sqrt{2} + 2\sqrt{2} - 4\sqrt{2} = -\sqrt{2}$.

If terms containing radicals cannot be collected in this manner, the addition or subtraction must be merely indicated.

Example 1. Perform indicated operation in $\sqrt{12} + \sqrt{24} + \sqrt{27} - \sqrt{54}$.

SOLUTION: $\sqrt{12} + \sqrt{24} + \sqrt{27} - \sqrt{54} = 2\sqrt{3} + 2\sqrt{6} + 3\sqrt{3} - 3\sqrt{6} = 5\sqrt{3} - \sqrt{6}$.

Example 2. Perform the indicated operations in $\sqrt[3]{16a^4b^5c^6} - \sqrt[3]{54ab^2c^3} - \sqrt[3]{x^4(a-b)^5} - \sqrt[3]{x^7(a-b)^2}$.

SOLUTION: This expression equals $2abc^2\sqrt[3]{2ab^2} - 3c\sqrt[3]{2ab^2}$
$- x(a-b)\sqrt[3]{x(a-b)^2} - x^2\sqrt[3]{x(a-b)^2}$
$= (2abc^2 - 3c)\sqrt[3]{2ab^2} - [x^2 + x(a-b)]\sqrt[3]{x(a-b)^2}$.

EXERCISES

Perform the following indicated operations.

1. $\sqrt{63} + \sqrt{32} - \sqrt{28} - \sqrt{50}$
2. $\sqrt{20} - \sqrt{45} + \sqrt{108} - \sqrt{48}$
3. $\sqrt[3]{16} - \sqrt[3]{81} + \sqrt[3]{54} - \sqrt[3]{375}$
4. $\sqrt[3]{a^5} + \sqrt[3]{8a^8} - \sqrt[3]{b^4} + \sqrt[3]{b^7}$
5. $\sqrt{16ab^3} - \sqrt{8c^2d} + \sqrt{4ab} - \sqrt{2d}$
6. $\sqrt[3]{7x^5m^3} - \sqrt[3]{9x^7m^4} + \sqrt[3]{56x^2} + \sqrt[3]{72x^4m}$
7. $\sqrt{8s^2t^2} + \sqrt{11a^2x} - \sqrt{25t} + \sqrt{44a^4x^3}$
8. $\sqrt[3]{m^5n^6} - \sqrt[3]{m^4n^5} - \sqrt[3]{m^7n^5} + \sqrt[3]{27m^2}$
9. $\sqrt{7a^4b^3} - \sqrt{36bc^3} - \sqrt{28a^6b^5} + \sqrt{49b^3c^3}$
10. $\sqrt[3]{r^2s^2t^5} + \sqrt[3]{l^2m^4n^6} + \sqrt[3]{r^5s^5t^2} - \sqrt[3]{8l^5m}$
11. $\sqrt{4(a-b)^3} + \sqrt{3ab^2c^3} - \sqrt{(a-b)^5} + \sqrt{12a^3c}$
12. $\sqrt[3]{6(x-y)^3} - \sqrt[3]{(x^2+y^2)^4} - \sqrt[3]{8(x^2+y^2)} + \sqrt[3]{48(x-y)^6}$
13. $\sqrt[3]{8(x^2-y^2)^4} + \sqrt[3]{6a^3bc^3} + \sqrt[3]{27(x^2-y^2)} - \sqrt[3]{6b}$
14. $\sqrt{16(x+y)^2} + \sqrt{x^2(a+b)} - \sqrt{9(a+b)^3} + \sqrt{c(a+b)^5}$
15. $\sqrt[3]{24(x+2a)} - \sqrt[3]{3(x+2a)^4} + \sqrt[3]{2(a-4b)} + \sqrt[3]{16(a-4b)}$
16. $\sqrt{(x^2-y^2)(x-y)} + \sqrt{(x^2-y^2)(x+y)^2} - \sqrt{(2x-y)^3}$
$+ \sqrt{(4x^2-y^2)(2x-y)}$

127. Multiplication of expressions containing radicals.—Multiplying monomials containing radicals depends on the two formulas at the right when m and n are rational fractions. To apply these rules it may be necessary to make certain changes that we have already studied.

$$a^m \cdot a^n = a^{m+n}$$
$$a^m \cdot b^m = (ab)^m$$

Thus, $\sqrt{2} \cdot \sqrt[3]{2} = 2^{1/2} \cdot 2^{1/3} = 2^{3/6+2/6} = 2^{5/6} = \sqrt[6]{2^5} = \sqrt[6]{32}.$
Again,
$\sqrt{2} \cdot \sqrt[3]{4} = 2^{1/2} \cdot 4^{1/3} = 2^{3/6} \cdot 4^{2/6} = 8^{1/6} \cdot 16^{1/6} = (8 \cdot 16)^{1/6} = 2\sqrt[6]{2}.$
These examples make use of the rule, $a^{p/q} = (a^p)^{1/q} = \sqrt[q]{a^p}$ and also of $a^{p/q} = a^{mp/mq}$.

EXERCISES

Express each of the following, using a single radical sign and no fractional exponents. Reduce each radical to the simplest form.

1. $\sqrt{2} \cdot \sqrt[3]{3}$ 2. $\sqrt[3]{2} \cdot \sqrt{2}$ 3. $\sqrt{2} \cdot \sqrt[3]{4}$ 4. $\sqrt{3} \cdot \sqrt[3]{3}$ 5. $\sqrt[3]{4} \cdot \sqrt{4}$
6. [illegible] 7. $5^{1/2} \cdot 3^{1/3}$ 8. $a^{1/2} \cdot a^{3/4}$ 9. $b^{2/3} \cdot b^{5/2}$ 10. $(2a^2)^{1/3}(4a^2)^{1/3}$
11. $\left(\frac{3}{4}\right)^{1/2}\left(\frac{3}{8}\right)^{1/2}$ 12. $\left(\frac{3}{5}\right)^{1/3}\left(\frac{9}{25}\right)^{1/3}$ 13. $\left(\frac{5}{6}\right)^{1/2}\left(\frac{2}{3}\right)^{1/2}$ 14. $\left(\frac{5}{8}\right)^{2/3}\left(\frac{3}{16}\right)^{1/3}$
15. $12^{1/2} \cdot \left(\frac{8}{45}\right)^{1/3}$ 16. $\dfrac{(a-b)^{1/3}\sqrt{(a+b)^3}}{a^2-b^2}$ 17. $\dfrac{(a+b)^{1/2}(a-b)^{1/2}}{\sqrt{a-b}}$
18. $x^{3/4}y^{3/4}$ 19. $\dfrac{(m-n)^{1/3}(m+n)^{1/3}}{(m^2-n^2)^{1/3}}$ 20. $\dfrac{(x^2-1)^{1/2}(x+1)^{1/2}}{x-1}$
21. $\dfrac{(x^2+1)^{2/3}(x+1)^{2/3}(x-1)^{2/3}}{(x^2+1)^{1/3}}$ 22. $2^{1/2} \cdot 3^{1/2} \cdot 4^{1/3} \cdot 5^{1/2}$
23. $(9x^3b^4c^5d^6)^{2/3}$ 24. $(3b^3c^4d^5)^{1/4}$

Write each of the following, using a single radical and no fractional exponents.

25. $4a^{1/2}b^{1/2}$	26. $3a^{1/2}b^{1/3}$	27. $2a^{2/3}b^{1/3}$	28. $2^{2/3}b^{1/2}$
29. $x^{1/2}y^{1/4}$	30. $a^{1/2}b^{3/4}$	31. $a^{1/2}b^{1/3}c^{1/4}$	32. $x^{1/2}y^{1/3}z^{1/4}$
33. $x^{1/2}y^{1/3}z^{2/3}$	34. $a^{1/2}b^{2/3}c^{1/3}$	35. $p^{2/3}q^{1/2}r^{3/4}$	36. $h^{5/6}k^{3/4}l^{2/3}$

Write each of the following using fractional exponents. Reduce each fractional exponent to the lowest terms.

37. $\sqrt{x^3y^4}$	38. $\sqrt[3]{xy^2}$	39. $\sqrt[3]{x^2y^3z^4}$	40. $\sqrt{a^3b^2c}$
41. $\sqrt{2xz^3}$	42. $\sqrt[3]{4a^2c}$	43. $\sqrt{xy^2z^3b^4}$	44. $\sqrt[3]{5yz^4}$
45. $\sqrt{2\sqrt{x}}$	46. $\sqrt{3\sqrt[3]{2a^2}}$	47. $\sqrt{5\sqrt[3]{4a^2y^3}}$	48. $2\sqrt[3]{3\sqrt{xya^3}}$

Example. Find the product of $\sqrt[3]{(a+b)^2(a-b)}$ and $\sqrt[3]{(a-b)^2(a+b)^2}$.

SOLUTION: $\sqrt[3]{(a+b)^2(a-b)} \cdot \sqrt[3]{(a+b)^2(a-b)^2} = \sqrt[3]{(a+b)^4(a-b)^3}$
$= (a-b)(a+b)\sqrt[3]{a+b} = (a^2-b^2)\sqrt[3]{a+b}.$

Such examples may also be solved by using fractional exponents.

Thus, $\sqrt{a-b} \cdot \sqrt{a^2-b^2} = (a-b)^{1/2}(a^2-b^2)^{1/2}$
$= (a-b)^{1/2}(a-b)^{1/2}(a+b)^{1/2} = (a-b)(a+b)^{1/2},$

and $\sqrt[3]{(a+b)^2(a-b)} \cdot \sqrt[3]{(a-b)^2(a+b)^2}$
$= (a+b)^{2/3}(a-b)^{1/3}(a-b)^{2/3}(a+b)^{2/3} = (a+b)^{4/3}(a-b)^1$
$= (a+b)^1(a+b)^{1/3}(a-b)^1 = (a+b)^1(a-b)^1(a+b)^{1/3}$
$= (a^2-b^2)(a+b)^{1/3}.$

In practice the exponent 1 is, of course, omitted.

Polynomials containing radicals are multiplied exactly as are other polynomials already studied.

Thus, $(\sqrt{a}-\sqrt{b}+1)(\sqrt{a}+\sqrt{b}-1) = a+\sqrt{ab}-\sqrt{a}-\sqrt{ab}-b$
$+\sqrt{b}+\sqrt{a}+\sqrt{b}-1 = a-b+2\sqrt{b}-1.$

Remember that $a^0 = 1$ for all values of a. (See page 124.)

Thus, $(x^m - x^{-m})^2 = x^{2m} - 2x^mx^{-m} + x^{-2m}$
$= x^{2m} - 2x^0 + x^{-2m} = x^{2m} - 2 + x^{-2m}.$

EXERCISES

Perform the following indicated operations.

1. $(2a^{-m} - 3a^m)^2$
2. $(3a^{-m} + 2a^m)(3a^{-m} - 2a^m)$
3. $(x^{2a} + x^{-3a})^2$
4. $(\sqrt{x} - 2\sqrt{z} + 3)(\sqrt{x} + 2\sqrt{z} - 3)$
5. $(\sqrt{6x} - \sqrt{5})(\sqrt{6x} + \sqrt{5})$
6. $(\sqrt{a} - \sqrt{b} + \sqrt{c})^2$
7. $(x+y)^{3a}(x+y)^a$
8. $(2\sqrt{x}+5\sqrt{y})^{m+n}(2\sqrt{x}+5\sqrt{y})$
9. $(x^{5a} - x^{7a})(x^{4a} + x^{5a})$
10. $(\sqrt{3}a^{-m} + 2b^{-m})^2$
11. $(1 - 3\sqrt{m} + 2\sqrt{n})^2$
12. $(\sqrt{5a} + 4\sqrt{b})^2(\sqrt{5a} - 4\sqrt{b})$
13. $(\sqrt{3x} - \sqrt{y} + \sqrt{z})(\sqrt{3x} - 2\sqrt{y} - 3\sqrt{z})$
14. $(\sqrt{3a} + \sqrt{2b} - \sqrt{c})(\sqrt{3a} - \sqrt{2b} + \sqrt{c})$
15. $(\sqrt{8x} - \sqrt{18y} + \sqrt{3})(\sqrt{8x} + \sqrt{18y} - \sqrt{3})$
16. $(x + \sqrt{x} + 1)(x - \sqrt{x} + 1)(x + \sqrt{x} + 1)$
17. $(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})(a^{1/3} - b^{1/3})$
18. $(a^{1/2} + b^{1/2})(a^{1/4} + b^{1/4})(a^{1/4} - b^{1/4})$
19. $(x^{2/3} - x^{1/3}y^{1/3} + y^{2/3})(x^{1/3} - y^{1/3})$
20. $(a^{2/3} + b^{2/3})(a^{4/3} - a^{2/3}b^{2/3} + b^{4/3})$

128. Reduction of expressions containing negative and zero exponents.—In a fraction a factor may be changed from the numerator to the denominator, or in the opposite direction, by changing the sign of its exponent. This depends on the identity $a^{-k} = \dfrac{1}{a^k}$.

$$a^{-k} = \frac{1}{a^k}$$
$$a^0 = 1$$

EXERCISES

Write the following as fractions with no negative exponents. Leave no radical signs in the answers. Note that $a^1 \cdot a^{-1} = a^0 = 1$.

1. $\dfrac{ab^2c^{-2}}{a^{-1}bc^{-1}}$
2. $\dfrac{(x-y)^{-1}(x^2-y^2)}{2x^{-1}y^{-1}}$
3. $\dfrac{m^2n^{-1}m^2}{\sqrt{mn}}$
4. $\dfrac{x^0y^{-1}z^2}{a^0y^2z^{-3}}$
5. $\dfrac{p^3q^{-2}r^0}{s^3q^4r^2}$
6. $\dfrac{l^{-2}m^3n^{-1}}{m^2n^3l^4}$
7. $\dfrac{(a+b)^n(a-b)^{-1}}{(a^2-b^2)^{-1}}$
8. $\dfrac{p^{-3}q^2r^{-2}}{p^{-1/2}\sqrt{q}\sqrt{r^3}}$
9. $\dfrac{x^{-1/2}y^2z}{x^{-3/2}y^{2/3}q^{-2}}$

Using signed exponents, write the following without using the fractional form. Leave no radical signs in the answers.

10. $\dfrac{(a+b)^3(a^2-b^2)}{(a-b)^2(a^2-ab+b^2)}$
11. $\dfrac{p^{-2}r^2c^{-1}}{p^{-4}r^{-1}c^{-2}}$
12. $\dfrac{a^{-1}b^2c^{-2}}{a^3b^{-2}c^{-1}}$
13. $\dfrac{r^3c^{-2}d^{-4}a^5}{r^4c^{-3}d^6a^7}$
14. $\dfrac{5^0b^{-2}c^2m^3}{10^0b^2c^4m^{-2}}$
15. $\dfrac{a^0b^3c^{-3}\sqrt{d}}{p^0q\sqrt{c}\,d^{3/4}}$
16. $\dfrac{m^{2/3}n^{3/4}p^{-4/5}}{m^{1/2}n^{-1/4}p^{-9/5}}$
17. $\dfrac{c^{-3/5}d^0b^{2/3}}{c^0d^{-4}b^{-5/3}}$
18. $\dfrac{8k^{-3}l^3m^{-2}}{4^0k^{-4}l^{-1}m^{-3}}$
19. $\dfrac{r^{1/2}s^{-2}t^{1/2}}{r^3s^2t^{3/2}}$
20. $\dfrac{u^0v^{-3}w^2}{u^{3/4}v^{-2}w^{1/2}}$
21. $\dfrac{a^{3/4}x^{-1/2}z^{1/3}}{a^{-1/2}x^{1/3}z^{-1/2}}$

129. Rationalizing factors.—For a binomial surd, an expression of the type $\sqrt[p]{a} \pm \sqrt[q]{b}$, a multiplier may be found which will make the product a rational expression. We shall consider square roots only.

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$$

Clearly, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$.

Hence the sum of two square roots may be rationalized by multiplying by the difference of these square roots. In this case $\sqrt{x}-\sqrt{y}$ is a rationalizing factor for $\sqrt{x}+\sqrt{y}$. Similarly $\sqrt{x}+\sqrt{y}$ is a rationalizing factor for $\sqrt{x}-\sqrt{y}$.

EXERCISES

Find the rationalizing factor for each of the following and give the rational product.

1. $\sqrt{a+1} - \sqrt{a-1}$
2. $\sqrt{4} + \sqrt{(x+8)^3}$
3. $\sqrt{m-1} + \sqrt{m+1}$
4. $\sqrt{x^3+1} - \sqrt{x^3-1}$
5. $\sqrt{x^3+1} + \sqrt{x^3-1}$
6. $\sqrt{x^2+4} - \sqrt{x+2}$
7. $\sqrt{18} - \sqrt{15}$
8. $\sqrt{21} + \sqrt{7}$
9. $a\sqrt{a+1} - b\sqrt{b-1}$
10. $\sqrt{6x} + \sqrt{7y}$
11. $a\sqrt{2x} + b\sqrt{y}$
12. $b\sqrt{a-1} + a\sqrt{b+1}$

130. Rationalizing denominators of fractions.—It is sometimes necessary to multiply both terms of a fraction by a factor that will rationalize the denominator. This multiplier must then be a rationalizing factor of the denominator.

Example 1. $\dfrac{k}{\sqrt{a}+\sqrt{b}} = \dfrac{k}{\sqrt{a}+\sqrt{b}} \cdot \dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{k(\sqrt{a}-\sqrt{b})}{a-b}$

Example 2. $\dfrac{k}{\sqrt{a}-\sqrt{b}} = \dfrac{k}{\sqrt{a}-\sqrt{b}} \cdot \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \dfrac{k(\sqrt{a}+\sqrt{b})}{a-b}$

This process is called rationalizing the denominator.

The rule is: If the denominator is of the form $\sqrt{x} + \sqrt{y}$, multiply both terms by $\sqrt{x} - \sqrt{y}$; and if it is of the form $\sqrt{x} - \sqrt{y}$, multiply both terms by $\sqrt{x} + \sqrt{y}$.

EXERCISES

Rationalize the denominator in each of the following.

1. $\dfrac{3}{7+\sqrt{5}}$
2. $\dfrac{8}{\sqrt{7}-\sqrt{3}}$
3. $\dfrac{6(a+b)}{\sqrt{a}-\sqrt{b}}$
4. $\dfrac{5(x+a)}{\sqrt{x}+\sqrt{a}}$
5. $\dfrac{5}{\sqrt{7}-2}$
6. $\dfrac{2}{\sqrt{3}+\sqrt{5}}$
7. $\dfrac{2x}{\sqrt{y}-\sqrt{z}}$
8. $\dfrac{8}{\sqrt{18}-\sqrt{15}}$
9. $\dfrac{4(m+2n)}{\sqrt{m}-\sqrt{2n}}$
10. $\dfrac{6a+b}{\sqrt{2a}+\sqrt{b}}$
11. $\dfrac{3a+1}{\sqrt{5a}-\sqrt{3}}$
12. $\dfrac{6m+5}{\sqrt{6}+\sqrt{7m}}$

131. Rationalizing numerators.—It is sometimes necessary to rationalize the numerator of a fraction, possibly leaving radicals in the denominator. The process is similar to that used in rationalizing denominators.

Thus, $\dfrac{\sqrt{a}+\sqrt{b}}{k}=\dfrac{\sqrt{a}+\sqrt{b}}{k}\cdot\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\dfrac{a-b}{k(\sqrt{a}-\sqrt{b})}.$

EXERCISES

Rationalize numerators in the following.

1. $\dfrac{a+\sqrt{7}}{6}$
2. $\dfrac{4-\sqrt{2}}{4+\sqrt{2}}$
3. $\dfrac{\sqrt{3}+\sqrt{2}}{5}$
4. $\dfrac{\sqrt{2a}+\sqrt{b}}{a+b}$
5. $\dfrac{a\sqrt{a}-\sqrt{b}}{\sqrt{a}+b\sqrt{b}}$
6. $\dfrac{\sqrt{6m}-\sqrt{8n}}{2m+4n}$
7. $\dfrac{5\sqrt{x}+4\sqrt{y}}{2x-3y}$
8. $\dfrac{\sqrt{7y}-\sqrt{2z}}{z+y}$
9. $\dfrac{a\sqrt{2x}+b\sqrt{y}}{\sqrt{2x}-\sqrt{y}}$
10. $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$
11. $\dfrac{2\sqrt{x-3y}+3\sqrt{x+y}}{\sqrt{x-3y}-\sqrt{x+y}}$
12. $\dfrac{a\sqrt{a^2-b}-b\sqrt{a-b^2}}{a\sqrt{a^2-b}+b\sqrt{a-b^2}}$

132. Equations containing radicals.—Equations in which the unknown is involved in a radical expression are called radical equations. The radical equations considered in elementary courses in algebra contain no radicals other than square roots.

In solving radical equations the square roots must be removed at some stage of the process by squaring both members of the equation. If the equation contains only one radical, the terms are first transposed so that one member of the equation consists of this radical only.

Example 1. Solve $\sqrt{x-7}=2$.

SOLUTION: Squaring both members,

$$x-7=4 \text{ or } x=11.$$

By substitution we find that this value of x satisfies the original equation and is therefore a solution.

Example 2. Solve $\sqrt{x^2+7}-x=1$.

SOLUTION: Transposing $-x$ and squaring both members we have

$$x^2+7=x^2+2x+1, \text{ or } x=3.$$

On substituting this is found to be a root of the original equation.

Example 3. Solve $\sqrt{x-3}+1=0$.
SOLUTION: Transposing and squaring we obtain

$$x-3=1 \text{ or } x=4.$$

On substituting we find that this is not a root of the original equation and hence we conclude that this equation has no solution. That this must be the case is at once evident from an inspection of the equation. In order that this equation shall be satisfied $\sqrt{x-3}$ must equal -1, which is not possible (see page 113). Hence it follows that any equation which reduces to the form $\sqrt{}+a=0$, where a is positive, has no solution.

That the formal steps of the solution given above lead to an impossible value of x is due to the fact that squaring $\sqrt{x-3}$ and $-\sqrt{x-3}$ leads to the same result. If the original equation were $-\sqrt{x-3}+1=0$, $x=4$ would be its solution.

Example 4. $\dfrac{x-5}{\sqrt{x-5}}=4.$

SOLUTION: Clearly the first member of this equation is equal to $\sqrt{x-5}$ since any expression divided by its square root is equal to its square root.

Hence, $$\sqrt{x-5}=4 \text{ and } x=21.$$

On substitution this proves to be a root.

Example 5. Solve the equation $\dfrac{x-9}{\sqrt{x}+3}=1.$

SOLUTION: Since $x-9=(\sqrt{x}+3)(\sqrt{x}-3)$, the first member of this equation equals $\sqrt{x}-3$ and hence $x=16$, which on substitution proves to be a root.

Example 6. Solve the equation $\dfrac{x-16}{\sqrt{x}-4}=8.$

SOLUTION: Solving, we have $x=16$. But substituting $x=16$ reduces the first member to the form $\frac{0}{0}$ and hence is not a solution.

From these examples it is clear that the results obtained by solving radical equations require special scrutiny. As in the case of ordinary fractional equations any value which reduces a denominator to zero must be rejected. Furthermore, other values obtained by solving such an equation may fail to satisfy it as is seen in Example 3 above.

EXERCISES

Solve the following equations and check each result.

1. $\dfrac{x-9}{\sqrt{x}-3}=7$
2. $\dfrac{x-25}{\sqrt{x}+5}=4$
3. $\sqrt{x^2+8}+4=x$
4. $\sqrt{x^2+5}=x-5$
5. $\sqrt{x-16}+8=1$
6. $\dfrac{x-49}{\sqrt{x}-7}=14$
7. $\sqrt{x^2+8}=x+2$
8. $\sqrt{2x-10}+6=0$
9. $\dfrac{x+1}{\sqrt{x+1}}=3$
10. $\sqrt{x^2+12}=x+6$
11. $\sqrt{x^2+3}=x+3$
12. $\dfrac{x-49}{\sqrt{x}-7}=1$
13. $\sqrt{x^2+3x}+4=x+1$
14. $\sqrt{x^2+5x}-3=x-1$
15. $\dfrac{2x-9}{\sqrt{2x}-3}=1$

133. Equations containing two radicals.—To solve an equation containing two radicals it may be necessary to remove one radical at a time by squaring the members of an equation.

Example 1. Solve $\sqrt{x+1}+\sqrt{x+2}=6$

SOLUTION: Squaring $x+1+2\sqrt{x^2+3x+2}+x+2=36$.

Transposing and collecting terms,

$$2\sqrt{x^2+3x+2}=-2x+33.$$

Squaring again, $4x^2+12x+8=4x^2-132x+1089$

$$144x=1081,\ x=7\frac{73}{144}$$

Example 2. Solve $\sqrt{2x+3}=\dfrac{3x-1}{\sqrt{3x}-1}$

SUGGESTION: Note that $3x-1=(\sqrt{3x}+1)(\sqrt{3x}-1)$.

This reduces the equation to $\sqrt{2x+3}=\sqrt{3x}+1$.

EXERCISES

Solve and check by going over the work again.

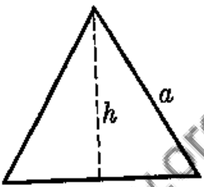
1. $\sqrt{x+1}+\sqrt{x+2}=3$
2. $\sqrt{x+2}+\sqrt{x+3}=4$
3. $1+\sqrt{x}=\sqrt{3+x}$
4. $\sqrt{5x-19}+\sqrt{3x+4}=9$
5. $\dfrac{\sqrt{4x+1}-\sqrt{3x-2}}{\sqrt{4x+1}+\sqrt{3x-2}}=\dfrac{1}{5}$
6. $\sqrt{5x}=-\dfrac{5x-1}{\sqrt{5x}+1}$
7. $\dfrac{x-a}{\sqrt{x}+\sqrt{a}}=\dfrac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a}$
8. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}$

PROBLEMS INVOLVING RADICALS

If the sides of a right triangle are a, b, and c, c being opposite the right angle, then $a^2 + b^2 = c^2$.

1. Solve $a^2 + b^2 = c^2$ for a and also for b. Notice that the length of a side of a triangle cannot be negative.

Translate the formulas obtained into rules stated in words.



2. The sides of an equilateral triangle are a; find the altitude and also the area in terms of a.

In all results found on this page rationalize denominators and remove square factors from under the radical sign.

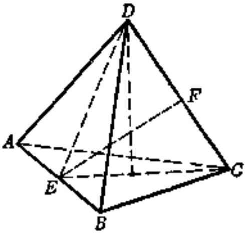
3. The area of an equilateral triangle is A. Find the length of its sides in terms of A.

4. The sides of a regular hexagon are equal to a; find the area in terms of a.

5. The area of a circle is A. ($A = \pi r^2$); express the radius in terms of A.

6. The sides of a triangle are $\frac{a}{2}\sqrt{3}$, $\frac{a}{2}\sqrt{3}$, a; find the altitude upon one of the sides $\frac{a}{2}\sqrt{3}$.

7. The edges of the triangular pyramid shown in the figure are all equal to a; find the lengths of DE and CE, E being the middle point of AB.



8. In the same figure find the altitude, h, of the pyramid.

SUGGESTION: Notice that the altitude of the pyramid is the altitude upon the side CE of the triangle DCE.

9. Find the volume of the pyramid in problem 7.

SUGGESTION: The volume of a pyramid is one-third the product of its base and altitude.

10. In the same figure F is the middle point of CD; find the length of EF.

11. If in the same figure $EF = c$, find a in terms of c. If $c = 8$, what is the value of a?

12. The inner and outer radii of a circular ring are r_1 and r_2 respectively. Express its area in terms of r_1^2 and r_2^2.

13. The inner radius of a circular ring is r, the outer radius is $1\frac{1}{2}r$, and the area A; express r in terms of A.

CHAPTER 10:

COMPLEX NUMBERS

From the standpoint of algebra we may regard the different kinds of numbers as added to our number system as they are needed for the purpose of solving different kinds of equations. In the following let a and b represent positive integers. Then, beginning with ordinary integers, the numbers that are needed are set opposite the equations to be solved.

	Kind of equation	*Additional numbers needed*
I	$x - a = 0$	None
II	$ax - b = 0$	Positive rational fractions
III	$x + a = 0$	Negative integers
IV	$ax + b = 0$	Negative rational numbers
V	$ax^2 - b = 0$	Irrational numbers
VI	$ax^2 + b = 0$	Imaginary numbers

The equation $ax^2 - b = 0$, or $ax^2 = b$, leads to the solution $x = \pm\sqrt{b/a}$, which includes such numbers as $\sqrt{2}$. The equation $ax^2 + b = 0$, or $ax^2 = -b$, leads to the solution $x = \pm\sqrt{-b/a}$. That is, to find a solution of this equation, we must have square roots of negative numbers. Since the square of a real number, positive or negative, is positive, it follows that we must have an entirely new kind of number.

134. Imaginary numbers.—To supply the need just mentioned, we shall now introduce a kind of number that has been "invented" for exactly this purpose. The unit of this number is indicated by $\sqrt{-1}$, which has the property that its square is -1.

The expressions $\sqrt{-1}$, $\sqrt{-4}$, $\sqrt{-6}$, $3\sqrt{-1}$ are called imaginary numbers, or pure imaginaries. The name is perhaps unfortunate, but it is now in general use and it is too late to consider a change of name.

135. Properties of the imaginary numbers.—The number $\sqrt{-1}$ is denoted by i (imaginary). Then $i^2 = -1$. The relations given at the right are easily found. Thus $i^3 = i^2 \cdot i = -1 \cdot i = -i$. $i^4 = i^3 \cdot i = -i \cdot i = -i^2 = 1$. $i^5 = i^4 \cdot i = i$, and so on.

$i^1 = i$	$i^5 = i$
$i^2 = -1$	$i^6 = -1$
$i^3 = -i$	$i^7 = -i$
$i^4 = 1$	$i^8 = 1$

We see, therefore, that i^n takes in order the values $i, -1, -i, 1$, as n takes the values 1, 2, 3, 4 then repeats these values as n increases from 5 to 8; then repeats them again as n increases from 9 to 12, and so on.

In the expression $\sqrt{-a}$ (a positive), $-a$ may be factored into a and -1, giving $\sqrt{a \cdot (-1)} = \sqrt{a}\sqrt{-1}$. The factor $\sqrt{a}$ is a real number and $\sqrt{-1}$ the imaginary unit. Hence any number of the type $\sqrt{-a}$ may be reduced to the product of a real number and the imaginary unit.

$$\sqrt{-a} = \sqrt{a}\sqrt{-1} = \sqrt{a}\, i$$
$$\sqrt{-4} = 2i$$
$$\sqrt{-7} = \sqrt{7}\, i$$

It is not a valid objection to the imaginary number to say that a negative number has no square root. This statement is true *if* we have only real numbers at our disposal; but it is not true if we have enlarged our number system so as to make it include imaginaries. The imaginary is simply another kind of number.

EXERCISES

Express each of the following as the product of i and a real number. It is to be understood that all letters have positive values. Reduce radicals to the simplest form.

1. $\sqrt{-4}$ 2. $\sqrt{-9}$ 3. $\sqrt{-16}$ 4. $\sqrt{-3}$ 5. $\sqrt{-5}$

6. $\sqrt{-a}$ 7. $\sqrt{-ab}$ 8. $\sqrt{-a^2b}$ 9. $\sqrt{-8}$ 10. $\sqrt{-12}$

11. $\sqrt{-a^2b^3}$ 12. $\sqrt{-18}$ 13. $\sqrt{-a^3b^4}$ 14. $\sqrt{-32a^2}$

15. $\sqrt{-(a+b)^3}$ 16. $\sqrt{-8a^3b^2}$ 17. $\sqrt{-5a^2b^4}$ 18. $\sqrt{-64(x+y)^4}$

19. $\sqrt{-\frac{1}{2}}$ 20. $\sqrt{-\frac{2}{3}}$ 21. $\sqrt{-\frac{3a^2b^3}{x^3b^4}}$ 22. $\sqrt{-\frac{a^2+b^2}{(a+b)^2}}$

23. $\sqrt{-\frac{3ab}{4m^3n^2}}$ 24. $\sqrt{-\frac{abc}{mn}}$ 25. $\sqrt{-\frac{4p^2r}{9m^2n}}$ 26. $\sqrt{-\frac{7(a+b)}{9(a^2-b^2)}}$

136. Complex numbers.—A combination like $2 + i$, or $3 - 4i$ is called a complex number. The general form of the complex number is $a + bi$, where a and b are any real numbers. The parts a and bi cannot be "added" in the sense of combining them into one number. In fact, the sign before bi is not a sign of addition but simply the sign of bi. That is, $a + bi$ is really a pair of numbers, one element being a and the other $+bi$. The letters a and b may, of course, be either positive or negative real numbers. That is, $1 + i$, $1 - i$, $-1 + i$, $-1 - i$, $6 + 5i$, $-9 + 2i$ are all complex numbers. If in $a + bi$, $a = 0$, then we have the pure imaginary bi and if $b = 0$, we have the real number a. Hence the real numbers a and also pure imaginaries bi are special cases of the complex number $a + bi$. In $a + bi$, a is called the real part and bi the imaginary part. A special property of complex numbers is that if two such numbers are equal, then their real parts and also their imaginary parts are equal.

$$a + bi$$

If $a + bi = c + di$, then $a = c$ and $b = d$

Addition

$$\begin{array}{c} a + bi \\ c + di \\ \hline a + c + (b + d)i \end{array}$$

Subtraction

$$\begin{array}{c} a + bi \\ c + di \\ \hline a - c + (b - d)i \end{array}$$

137. Addition and subtraction of complex numbers.—In adding and subtracting complex numbers, the real and the imaginary parts are added or subtracted separately.

Thus $(a + bi) + (c + di) = a + c + (b + d)i$ and $(a + bi) - (c + di) = a - c + (b - d)i$.

Note that $bi + di = (b + d)i$ and $bi - di = (b - d)i$.

That is, b and d are treated as the coefficients of i exactly as if i were a real number.

EXERCISES

In each of the following add the two complex numbers. Then subtract the lower number from the one above it.

1. $\begin{array}{c} 4 + 3i \\ \underline{2 - 2i} \end{array}$
3. $\begin{array}{c} 10 + 3i \\ \underline{-7 + 4i} \end{array}$
5. $\begin{array}{c} 6 - 2i \\ \underline{2 - 4i} \end{array}$
7. $\begin{array}{c} 10 - 8i \\ \underline{12 + 2i} \end{array}$
2. $\begin{array}{c} -3 + 5i \\ \underline{8 - 4i} \end{array}$
4. $\begin{array}{c} 6 - 4i \\ \underline{8 - 9i} \end{array}$
6. $\begin{array}{c} -3 + 2i \\ \underline{4 - 5i} \end{array}$
8. $\begin{array}{c} 1 + i \\ \underline{1 - i} \end{array}$

138. Multiplication of complex numbers.—The complex numbers $a + bi$ and $c + di$ are multiplied like any two binomials as shown at the right. Note that $bd(i)^2 = bd(-1) = -bd$.

$$\begin{array}{l} a + bi \\ c + di \\ \hline ac + cbi \\ \qquad adi + bd(i)^2 \\ \hline ac - bd + (cb + ad)i \\ = P + Qi, \quad \text{where} \\ P = ac - bd, \quad Q = cb + ad \end{array}$$

EXERCISES

Express each of the following as a real number or as a pure imaginary. Note that the general form of the result is $P + Qi$.

1. $(3ai)^2$
2. $2i(3i)^2$
3. $(2ai)(bi)^3$
4. $5x^2i(2y^2i)$
5. $(abi)^3$
6. $p^2i \cdot q^2i^3$
7. $im^2 \cdot i^3n^2$
8. $ri \cdot mi \cdot ni \cdot pi$

Multiply. Reduce products to the form $P + Qi$.

9. $\begin{array}{r} 1 + i \\ \underline{1 - i} \end{array}$

10. $\begin{array}{r} 3 - 2i \\ \underline{2 + 3i} \end{array}$

11. $\begin{array}{r} 2 + 4i \\ \underline{1 - 3i} \end{array}$

12. $\begin{array}{r} 4 + 2i \\ \underline{-2 + 3i} \end{array}$

13. $\begin{array}{r} 8 - 4i \\ \underline{2 + 3i} \end{array}$

14. $\begin{array}{r} -7 + 8i \\ \underline{3 - 6i} \end{array}$

15. $\begin{array}{r} a + bi \\ \underline{a + bi} \end{array}$

16. $\begin{array}{r} a - bi \\ \underline{a - bi} \end{array}$

17. $\begin{array}{r} a + bi \\ \underline{a - bi} \end{array}$

18. $\begin{array}{r} 2x + 5i \\ \underline{x - 1 - 4i} \end{array}$

19. $\begin{array}{r} a^2 + b^2i \\ \underline{a^2 - b^2i} \end{array}$

20. $\begin{array}{r} 3x + 2yi \\ \underline{4x - 5yi} \end{array}$

139. Division of complex numbers.—To divide a complex number $a + bi$ by a complex number $c + di$, indicate the division as a fraction and then simply the fraction. We note that $(c + di)(c - di) = c^2 - (di)^2 = c^2 + d^2$. Then we have:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

The quotient thus reduces to a complex number, of which $\dfrac{ac + bd}{c^2 + d^2}$ is the real part and $\dfrac{bc - ad}{c^2 + d^2}i$ is the imaginary part. Hence the quotient of any two complex numbers is of the form $P + Qi$, where P and Q are real numbers. Note that $c + di$ is a rationalizing factor of $c - di$ and $c - di$ is a rationalizing factor of $c + di$.

EXERCISES

Reduce each of the following fractions to the form $P + Qi$. Note that $\frac{a}{bi} = \frac{a}{bi} \cdot \frac{i}{i} = \frac{ai}{-b} = -\frac{a}{b}i$. Note also that i is a rationalizing factor of i^3.

1. $\frac{3}{2i}$ 2. $\frac{2i}{3i}$ 3. $\frac{4i}{5i^2}$ 4. $\frac{a}{bi^3}$ 5. $\frac{pi^3}{qi^2}$ 6. $\frac{ai \cdot bi}{ci^3}$

7. $\frac{1+i}{1-i}$ 8. $\frac{1-i}{1+i}$ 9. $\frac{3-i}{2+i}$ 10. $\frac{2+5i}{3-2i}$

11. $\frac{2+3i}{3-2i}$ 12. $\frac{a-2i}{b+5i}$ 13. $\frac{a+bi}{c+4i}$ 14. $\frac{x^2-y^2i}{x+yi}$

15. $\frac{a+b+ci}{a-b-ci}$ 16. $\frac{2x-3+yi}{x+3-yi}$ 17. $\frac{x^2-1+2yi}{x^2+1-2yi}$

140. The algebraic number system.—Beginning with the ordinary positive integers, our system of numbers has been extended by adding to it in order, (a) ordinary rational fractions, (b) irrational numbers, (c) negative numbers, (d) complex numbers.

$a + bi$	(1)
a	(2)
bi	(3)

In the general expression $a + bi$, a and b are any real numbers, integral or fractional, irrational, positive or negative. If in this expression $b = 0$, the number reduces to the real number a. If $a = 0$, it reduces to the pure imaginary, bi. All the laws of arithmetic (see page 17) hold for this extended number system. In particular, the sum, difference, product, and quotient of any complex numbers are unique complex numbers. That is, performing any of these operations on any two numbers of the system gives a result $P + Qi$, which is in this system.

Further, taking any root of any number in this system also gives a result within this system. However, this last operation does not give a unique result.

In the complex number system any number has two square roots, three cube roots, and in general n nth roots.

The real unit, 1, the negative unit, -1, and the imaginary unit i have interesting properties related to multiplication:

1. Every power of 1 is 1.
2. Powers of -1 are alternately -1 and 1.
3. Powers of i go in a cycle of i, -1, $-i$, 1. (See page 131.)

EXERCISES

1. Simplify the fraction $\frac{3+2i}{2-3i}$, as shown on page 133, §139.

Reduce each of the following to the form $P + Qi$.

2. $\frac{4-3i}{1+i}$
3. $\frac{7+2\sqrt{-1}}{1-\sqrt{-1}}$
4. $\frac{5-3i}{5+3i}$
5. $\frac{2-4\sqrt{-1}}{3+2\sqrt{-1}}$
6. $\frac{8+9i}{3-7i}$
7. $\frac{18-7\sqrt{-1}}{3+2\sqrt{-1}}$
8. $\frac{9}{3-7i}$
9. $\frac{4i}{6+2i}$
10. $\frac{3\sqrt{-1}}{8-3\sqrt{-1}}$

In the following perform the indicated operations. Reduce each result to the form $P + Qi$.

11. $(8+3i)+5-8i$
12. $(3x+y\sqrt{-1})+(2x-y\sqrt{-1})$
13. $(3a+2bi)+(a-bi)$
14. $(9i-7x)+(4-6i)$
15. $(5+7i)-(2-3i)$
16. $(4x-3+3i)-(2x+4-2i)$
17. $9a+3b+3i-(2a-3bi)$
18. $(8m-4mi+2)-(8-9mi)$
19. $(8-3x)(6+3i)$
20. $(14i-9)(3i+4)$
21. $(3-4i)\div(6+2i)$
22. $(3+2i)\div(5-2i)$
23. $(a+3i)(a-3i)$
24. $(a+bi)(a-bi)$
25. $(a+bi)^2$
26. $(a-bi)^2$
27. $(a+bi)^2+3(a+bi)$
28. $[(a+bi)^2-2(a+b)]-(a-bi)$
29. $2i(6-3i)(6+3i)$
30. $5i(2+3i)\div(2-3i)$
31. $(\sqrt{-1}+3)^2\div(\sqrt{-1}+1)$
32. $2\sqrt{-1}(1-\sqrt{-1})\div(1+\sqrt{i})$
33. $3\sqrt{-1}(2+5\sqrt{-1})\div\sqrt{-1}$
34. $(i-1)(i+1)\div(i-2)$
35. $(i-1)^2(i+1)^2$
36. $(a+bi)^3$
37. $(a-bi)^3$
38. $\left(\frac{-1-\sqrt{3}i}{2}\right)^3$
39. $\left(\frac{-1+\sqrt{3}i}{2}\right)^3$
40. $\left(\frac{-1+\sqrt{3}i}{2}\right)^2$
41. $\left(\frac{-1-\sqrt{3}i}{2}\right)^2$
42. $\left(\frac{-1-\sqrt{3}i}{2}\right)^2\left(\frac{-1+\sqrt{3}i}{2}\right)$
43. $\frac{-1-\sqrt{3}i}{2}\cdot\left(\frac{-1+\sqrt{3}i}{2}\right)^2$

44. Using the results in examples 38 and 39, how many solutions can you find of the equation $x^3 - 1 = 0$?

45. Solve the equation $x^3 + 1 = 0$ by factoring, $(x+1)(x^2-x+1) = 0$. Then solve $x^2 - x + 1 = 0$.

46. Find four solutions of $x^4 - 1 = 0$. Check your results by substitution in the given equation.

47. Find six solutions of $x^6 - 1 = 0$. Check by substituting in the given equation.

141. Further study of the system of algebraic numbers.—This page contains some repetition. In the sequence of classes of numbers given at the right, each class contains all the numbers in the class that precedes it. Thus the set of positive rational numbers contains all positive integers. Class B is obtained by adding to Class A all positive rational fractions.

Positive integers	A
Positive rational numbers	B
Positive real numbers	C
Signed numbers	D
Complex numbers	E

Class C is obtained by adding to Class B all positive irrational numbers. Class D is obtained by adding to Class C all negative real numbers, and Class E is obtained by adding to D all numbers $P + Qi$ in which Q is not zero. Thus we see that the system $P + Qi$ contains all numbers used in algebra.

As on page 17, the possibility of carrying out the four fundamental operations of arithmetic on the complete system of algebraic numbers and the uniqueness of the results are indicated by the four equalities at the right.

$$a + b = s$$
$$a - b = d$$
$$ab = p$$
$$a \div b = q$$

We now assume that the laws stated on page 17 hold for all numbers in the complete Class E. These laws will be regarded as the axioms of the system of algebraic numbers.

EXERCISES

1. If we start with numbers in Class A (positive integers), for which of these operations do these four results s, d, p, q lie within that class? What restrictions, if any, are necessary?

2. If we start as in example 1, within what class are we certain to find all the results? Discuss fully for each operation.

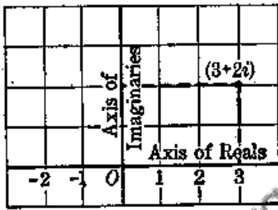
3. Discuss as in example 2, if we start with any number in Class B, in Class C, in Class D. Does any of the four operations lead to numbers that are not contained in Class D?

4. What operation of algebra may lead from Class A to Class E? from Class B to Class E? from Class C to Class E? from Class D to Class E?

5. If we start with a first degree equation $ax + b = 0$ with a and b real numbers, in which class of numbers will the solution be found?

6. In which class of numbers will the solution of the equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ be found if all the constants are real numbers?

142. Geometric representation of complex numbers.—By adopting a very simple device it is possible to represent the points in a plane by complex numbers $a + bi$. As in the case of the usual system of coordinates, consider two lines at right angles to each other, meeting in a point O called the origin. The horizontal line is now called the axis of reals, and the vertical line the axis of imaginaries. These are also called the real axis and the imaginary axis.



On the real axis positive and negative numbers a are laid off exactly as in the Cartesian system. The real numbers b in bi are laid off on the imaginary axis. The symbol i is thus made to indicate that the b is laid off on the vertical axis. That is, it indicates a direction at right angles to the real axis.

The complex number $3 + 2i$ is made to represent the point 3 units to the right of the imaginary axis and 2 units above the real axis. The Cartesian coordinates of this point are 3, 2.

In general, the complex number $a + bi$ represents the point a distance a from the imaginary axis and a distance b from the real axis. The directions from these axes depend upon the signs a and b.

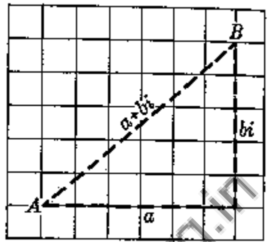
When pairs of real numbers are used to represent the points of a plane, we speak of the plane as the Cartesian plane; and when one complex number is used to represent each point in a plane, we speak of the plane as the complex plane. No contradiction is involved here since each arrangement is purely arbitrary. The use of the complex plane is often highly convenient both in mathematical theory and in practical applications.

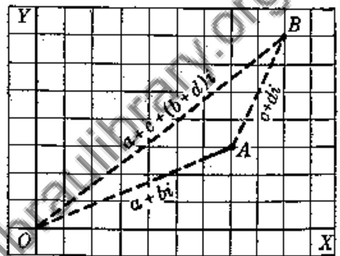
EXERCISES

Find the Cartesian coordinates of the points represented by the following.

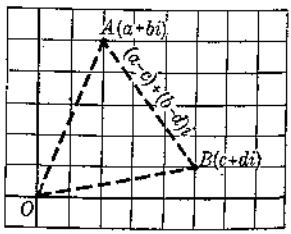
1. $7 - 3i$ 2. $3 + 9i$ 3. $-4 + 2i$ 4. $-3 - 5i$ 5. $-3/4 + 4/5i$
6. $(2 + 3i) + (4 - 5i)$ 7. $(2 + 5i)(4 - 5i)$ 8. $(1 - 9i)(4 + i)$
9. $(3 - i) \div (2 + i)$ 10. $(5 + 2i) \div (1 - 3i)$ 11. $(4 - i) \div (2 + 7i)$
12. $\dfrac{1 + 2i}{1 - i}$ 13. $\dfrac{1 + 2i}{1 + i}$ 14. $\dfrac{1 - 2i}{1 - i}$ 15. $\dfrac{1 - 2i}{1 + i}$
16. $\dfrac{3 - 3i}{1 - i}$ 17. $\dfrac{2 - 2i}{i - 1}$ 18. $\dfrac{1 + 2i}{3 + 4i}$ 19. $\dfrac{1 - 2i}{3 + 4i}$

143. A complex number regarded as a vector.—A complex number may also be regarded as representing a line segment starting at any point A in the plane, and having a length and direction determined by going a distance and direction a in the direction of reals and a distance and direction b in the imaginary direction. The directed segment AB, $a + bi$ as shown in the figure, is called a <u>vector.</u> In this sense every complex number represents a vector.





144. Vector addition and subtraction.—The addition of the two vectors $a + bi$ and $c + di$ is shown at the right. The first vector, $a + bi$, begins at the origin and ends at A. The second vector, $c + di$, begins at A and ends at B. The sum of the two vectors is the segment OB represented by the complex number $a + c + (b + d)i$. Therefore the addition of complex numbers may be regarded as vector addition.



To subtract $c + di$ from $a + bi$ we must find a vector which added to the subtrahend $c + di$ gives the minuend $a + bi$ as the sum. Using the figure, we must find the vector BA, since in vector addition $OA + AB = OB$. Clearly $a - c + (b - d)i$ represents the vector BA.

EXERCISES

Represent the following as vector additions and subtractions. Construct a figure for each example.

1. $(3 + 2i) + (6 + 4i)$
2. $(2 + 5i) + (6 - 3i)$
3. $(4 - 3i) + (2 + 3i)$
4. $(5 - 3i) + (3 - 2i)$
5. $(8 - 3i) + (-2 + i)$
6. $(5 - 7i) + (5 + 7i)$
7. $(6 + 2i) - (3 + i)$
8. $(5 + 3i) - (4 + 5i)$
9. $(3 + 7i) - (-8 + 3i)$
10. $(-1 + i) + (1 - i)$
11. $(-2 - 3i) - (5 - 6i)$
12. $(7 + 2i) - (-1 - 5i)$

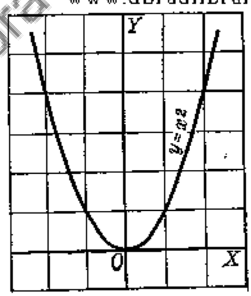
CHAPTER 11:

QUADRATIC EQUATIONS

An equation in which the unknown occurs in the second degree but not in a higher degree is called a quadratic equation. The general form of this equation is $ax^2 + bx + c = 0$. For the purpose of studying this equation we shall begin with a study of the graph of equations of the type $y = ax^2 + bx + c$. It turns out that if a is fixed, then all graphs of such equations are of the same "shape." We therefore begin with the special case when $b = 0$ and $c = 0$.

145. The parabola.—The equation $y = x^2$ is satisfied by the following pairs of numbers.

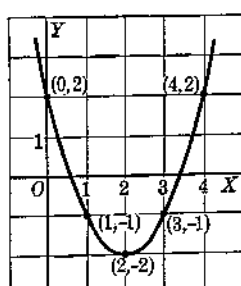
x	0	1	2	3	−1	−2	−3	½	−½	¼	−¼
y	0	1	4	9	1	4	9	¼	¼	$\frac{1}{16}$	$\frac{1}{16}$



If we plot the points of which these pairs of numbers are the coordinates and then draw a smooth curve through them, we have the curve shown at the right, which is a parabola.

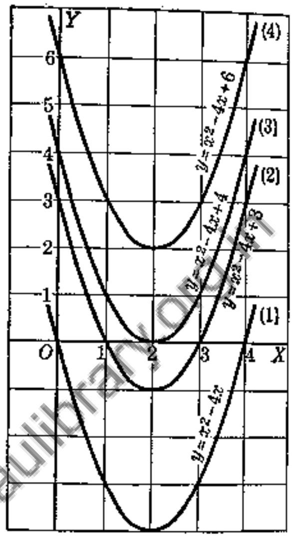
To construct the graph of $y = x^2 - 4x + 2$ we have,

x	0	1	2	3	4	5	−1
y	2	−1	−2	−1	2	7	7



From these points we obtain the second parabola at the right. If we study these curves we shall find that they are exactly of the same shape.

146. Intersections of parabolas and the x-axis.—In the figure we have the parabolas represented by the equations $y = x^2 - 4x$, $y = x^2 - 4x + 3$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$.



We shall now attempt to find the points in which these curves meet the x-axis. The y-coordinates of the points in which the curves meet this axis are 0. Hence we put $y = 0$ in each equation:

$$\begin{aligned} &(1)\ x^2 - 4x = 0 \\ &(2)\ x^2 - 4x + 3 = 0 \\ &(3)\ x^2 - 4x + 4 = 0 \\ &(4)\ x^2 - 4x + 6 = 0 \end{aligned}$$

Example 1. Solve the equation $x^2 - 4x = 0$. Clearly, $x = 0$ is a root of this equation since $0^2 - 4 \cdot 0 = 0$. But $x = 4$ is also a root since $4^2 - 4 \cdot 4 = 0$. The roots of this equation are seen most easily if the left member is written in the factored form $x(x - 4) = 0$. A value of x that makes either factor zero will make the whole product zero. That is, $x = 0$ and $x = 4$ both make the left member zero, and hence satisfy the equation. Hence the curve (1) meets the x-axis in the points (0, 0) and (4, 0).

$$\begin{aligned} x^2 - 4x &= 0 \\ x(x - 4) &= 0 \\ x = 0, \quad x &= 4 \end{aligned}$$

Example 2. Solve the equation $x^2 - 4x + 3 = 0$. When the left member is written in the factored form as shown at the right, it is clear that $x = 1$ is a root since that makes the first factor, $x - 1$, equal to zero. Also $x = 3$ is a root since that makes the second factor, $x - 3$, equal to zero. Hence curve (2) meets the x-axis in the points (1, 0), (3, 0).

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x - 1)(x - 3) &= 0 \\ x = 1, \quad x &= 3 \end{aligned}$$

Example 3. Solve the equation $x^2 - 4x + 4 = 0$. When the left member is factored it is evident that $x = 2$ is a root since that makes both factors equal to zero. Hence curve (3) meets the x-axis in the point (2, 0) and in no other point.

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x - 2)(x - 2) &= 0 \\ x &= 2 \end{aligned}$$

Example 4. Solve the equation $x^2 - 4x + 6 = 0$. From the graph it is evident that curve (4) does not meet the x-axis. If we attempt to solve the equation $x^2 - 4x + 6 = 0$ we encounter some trouble. Later, when we solve this equation, we shall find that the values of x are complex numbers. That is, there are no real numbers that satisfy this equation. This solution means that the curve $y = x^2 - 4x + 6$ does not meet the x-axis.

$$x^2 - 4x + 6 = 0$$
$$x = 2 + \sqrt{-2}$$
$$x = 2 - \sqrt{-2}$$

EXERCISES

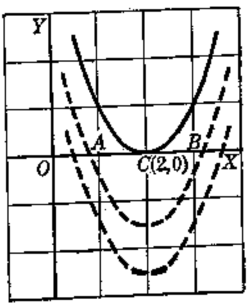
Find the points in which each of the following curves cuts the x-axis. In each example put $y = 0$ and then solve the resulting equation in x. These quadratics may be solved by factoring as above.

1. $y = x^2 - 3x + 2$
2. $y = x^2 + 3x + 2$
3. $y = x^2 - 5x + 6$
4. $y = x^2 + 5x + 6$
5. $y = x^2 - x - 30$
6. $y = x^2 + 6x + 9$
7. $y = x^2 - 6x + 9$
8. $y = x^2 - 2x - 3$
9. $y = x^2 - 4x - 12$
10. $y = x^2 - 6x$
11. $y = x^2 + 6x$
12. $y = x^2 - ax$
13. $y = x^2 + ax$
14. $y = x^2 + 2ax + a^2$
15. $y = x^2 - 2kx + k^2$
16. $y = x^2 - 8x + 12$
17. $y = x^2 + 9x - 22$
18. $y = x^2 - 13x + 22$

147. *Double roots.*—In solving the equation $x^2 - 4x + 4 = 0$ we found that it is satisfied by $x = 2$, and by no other number. However, there are two factors $x - 2$, and hence we may think of 2 as being a repeated or <u>double root</u>. This way of regarding this root is, in fact, the general usage.

$$x^2 - 4x + 4 = 0$$
$$(x - 2)(x - 2) = 0$$
$$x = 2, \quad x = 2$$

Let us consider the parabola $y = x^2 - 4x + c$ for different values of c. If c is less than 4, then the parabola meets the x-axis in two distinct points A and B. If we let c grow larger, approaching 4, then the points A and B come closer and closer together; and the moment that c becomes 4, these points coincide at the point $C(2, 0)$ in the figure. That is, the equation has two <u>coincident roots</u>, or one double root when $c = 4$. When the left member of the equation $ax^2 + bx + c = 0$ is a perfect square we always have a double root of this type.



148. Solving a quadratic by completing the square.—Consider the equation $x^2 - 6x + 4 = 0$. By transposing 4 to the second member and then adding 9 to both members, we obtain equation (3) at the right, in which the left member is a perfect square. Taking square roots of both members results in equation (5) and the solution given in (6).

$$
\begin{aligned}
x^2 - 6x + 4 &= 0 \quad (1)\\
x^2 - 6x &= -4 \quad (2)\\
x^2 - 6x + 9 &= 5 \quad (3)\\
(x - 3)^2 &= 5 \quad (4)\\
x - 3 &= \pm\sqrt{5} \quad (5)\\
x &= 3 \pm \sqrt{5} \quad (6)
\end{aligned}
$$

The purpose in adding 9 to both members (and not some other number) is to make the left member a perfect square (see p. 40).

In an equation like $3x^2 - 7x - 5 = 0$, this "completing of the square" is done most easily by first multiplying by 3, so as to make the first term a square, $9x^2$. Then the steps are as shown at the right. The solution of this type of equation is given in a more general form in the next section.

$$
\begin{aligned}
3x^2 - 7x - 5 &= 0\\
3x^2 - 7x &= 5\\
9x^2 - 21x &= 15\\
9x^2 - 21x + \left(\frac{7}{2}\right)^2 &= 15 + \left(\frac{7}{2}\right)^2\\
&= \frac{109}{4}\\
3x - \frac{7}{2} &= \pm\sqrt{\frac{109}{4}}\\
&= \pm 1/2\sqrt{109}\\
x &= \frac{7/2 \pm 1/2\sqrt{109}}{3}\\
&= \tfrac{1}{6}(7 \pm \sqrt{109})
\end{aligned}
$$

Following is a rule for completing the square.

1. Multiply the members of the equation by a number that will make the first term a square.
2. Divide the second term by twice the square root of this square term, and then square the quotient.
3. Add this last square to both members of the equation.

EXERCISES

1. Show that $9x^2 - 21 + (7/2)^2$ is a trinomial square.

Solve the following equations.

2. $x^2 - 4x + 2 = 0$
3. $x^2 + 8x - 9 = 0$
4. $x^2 + 8x + 4 = 0$
5. $x^2 - 7x + 12 = 0$
6. $x^2 + 5x = 8$
7. $2x^2 - 3x + 4 = 0$
8. $2x^2 + 3x = 7$
9. $3x^2 + 5x = 14$
10. $3x^2 - 4x = 7$
11. $4x^2 + 7x = 11$
12. $4x^2 - 5x + 7 = 0$
13. $5x^2 + 9x = 15$
14. $6x^2 + 3x = 12$
15. $7x^2 - 6x = 15$
16. $9x^2 + 6x = 10$

149. General solution of the quadratic.—The method of solving equations by completing the square, which we have just studied, will now be used in solving the general quadratic equation $ax^2 + bx + c = 0$. The steps are shown at the right.

$$ax^2 + bx + c = 0 \quad (1)$$
$$ax^2 + bx = -c \quad (2)$$
$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac \quad (3)$$
$$(2ax + b)^2 = b^2 - 4ac \quad (4)$$
$$2ax + b = \pm\sqrt{b^2 - 4ac} \quad (5)$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (6)$$

Step 1. Transpose c.

Step 2. Multiply both members by $4a$ and add b^2 to each member.

Step 3. Note that the first member is now the square of $2ax + b$.

Step 4. Take square roots of both members.

Step 5. Transpose b and divide by $2a$.

Since every quadratic can be reduced to the form $ax^2 + bx + c = 0$, it follows that we have a formula in which we may substitute to find the roots of any quadratic. The two values of x are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

In passing from (4) to (5) above we take the positive square root of the first member and the two square roots of the second member. If we were to take both square roots of the first member also, we should have

$$2ax + b = +\sqrt{b^2 - 4ac} \quad (1) \qquad 2ax + b = -\sqrt{b^2 - 4ac} \quad (3)$$
$$-2ax - b = -\sqrt{b^2 - 4ac} \quad (2) \qquad -2ax - b = +\sqrt{b^2 - 4ac} \quad (4)$$

which really gives only two solutions since (1) and (2) are equivalent, as are also (3) and (4).

Example 1. Solve $x^2 - 4x + 3 = 0$. On page 140 we found that $x = 1$ and $x = 3$ are the roots of $x^2 - 4x + 3 = 0$. In this equation $a = 1$, $b = -4$, $c = 3$. Hence,

$$x_1 = \frac{-(-4) + \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{4 + \sqrt{16 - 12}}{2} = \frac{4 + 2}{2} = 3$$
$$x_2 = \frac{-(-4) - \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{4 - \sqrt{16 - 12}}{2} = \frac{4 - 2}{2} = 1$$

The solutions of the following Examples are shown at the right. Check these solutions by repeating the work.

Example 2. Solve $x^2 - 4x + 4 = 0$. Substituting in the formula, we find that the expression under the radical sign is 0 and hence we have the double root $x = 2, 2$. Compare page 141, §147.

$$x^2 - 4x + 4 = 0$$
$$x = \frac{4 \pm \sqrt{16 - 16}}{2}$$
$$= \frac{4 \pm 0}{2} = 2, 2$$

Example 3. Solve $x^2 - 4x = 0$. Note that in this case the c of the formula is zero. Compare solution on page 140, Example 1.

$$x^2 - 4x = 0$$
$$x = \frac{4 \pm \sqrt{4^2 - 0}}{2}$$
$$= \frac{4 \pm 4}{2} = 4, 0$$

Example 4. Solve $x^2 - 4x + 6 = 0$. As stated on page 141, Example 4, the solution gives two complex numbers. Note that $\sqrt{-8} = \sqrt{8} \cdot \sqrt{-1} = 2\sqrt{2}i$.

$$x^2 - 4x + 6 = 0$$
$$x = \frac{4 \pm \sqrt{16 - 24}}{2}$$
$$= 2 \pm \sqrt{-2}$$
$$\text{or} \quad 2 \pm \sqrt{2}i$$

EXERCISES

Using the general formula for the solution of a quadratic find the solutions of the following.

If the given equation is not in the standard form $ax^2 + bx + c = 0$, the first step before applying the formula is to reduce the equation to this form.

1. $3x^2 - 7x = 10$
2. $2s^2 - 3s + 5 = 0$
3. $2y^2 + 6y = 15$
4. $3y^2 - 9y = 14$
5. $7x^2 + 8x + 9 = 0$
6. $5y^2 - 12y = 12$
7. $9x^2 - 3x = 8$
8. $4x^2 - 5x = 2$
9. $x^2 - 4mx + m^2 = 0$
10. $4x^2 - 2x - 7 = 0$
11. $3x^2 - 2px + q = 0$
12. $w^2 - 2w + 5 = 0$
13. $5y^2 + 9y + 3 = 0$
14. $5x^2 - 3 = 2x^2 + 7$
15. $gx^2 + ax - s = 0$
16. $kx^2 + 2kx + x + 2 = 0$
17. $2x^2 - 5x + 2k = 0$
18. $kx^2 - 3x + 4 = 0$
19. $9p^2x^2 + 6px + 1 = 0$
20. $y^2 - (m + n)y + mn = 0$
21. $x^2 - 2\sqrt{3}x + 2 = 0$
22. $x^2 - x/a - 3/(4a^2) = 0$
23. $\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2} - \dfrac{24}{x^2-4} = 0$
24. $(p + x)(q - x) - (x - p)(q + x) = 0$
25. $(2x + 3)(3x - 2) = a^2 + a(5a + 3)$
26. $x^2 + \dfrac{a(a+b)}{3} = ax + \dfrac{(a+b)x}{3}$
27. $x^2 + 2a^2 + 3a - 2 = (3a + 1)x$

150. ***Relation between the coefficients and roots of a quadratic.***— The roots of the equation $x^2 - 5x + 6 = (x-2)(x-3) = 0$ are 2 and 3. The product of these roots is 6 and their sum is 5. This is a special case of the general theorem:

$$x^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$
$$x_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}$$
$$x_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$
$$x_1 + x_2 = -b$$
$$x_1x_2 = c$$

The product of the roots of the quadratic $x^2 + bx + c = 0$ *is* c *and their sum is* $-$b.

Prove this theorem by showing that the statements at the right are correct.

To solve the equation $x^2 + bx + c = 0$, therefore, is equivalent to finding two numbers whose sum is $-b$ and whose product is c.

If the equation is in the form $ax^2 + bx + c = 0$, the above form is obtained by dividing by a, giving $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

In this equation the product of the roots is c/a and their sum is $-b/a$.

The roots found in solving a quadratic may be checked by using the above theorem.

Using this theorem a quadratic equation may be formed whose roots will be two given numbers. The form at the right shows the process. The sum of the roots r_1 and r_2, with the sign changed is the coefficient of x, and the product of the roots, r_1r_2, is the constant term.

Roots: r_1, r_2
$$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

EXERCISES

Solve and check by using the theorem in §150.

1. $x^2 + 7x + 28 = 0$
2. $x^2 - 5x + 4 = 0$
3. $x^2 - 5x + 3 = 0$
4. $x^2 - 3ax + b = 0$
5. $x^2 - 3bx - a = 0$
6. $x^2 + mx + 4n = 0$
7. $x^2 - 19x + 4 = 0$
8. $x^2 + 9x - 8 = 0$
9. $x^2 - 8x + 16 = 0$
10. $2x^2 - 4x + 7 = 0$
11. $2x^2 + 5x - 8 = 0$
12. $4x^2 - 6x + 15 = 0$
13. $ax^2 + 4mx + m^2 = 0$
14. $(a+1)x^2 + (a-1)x + 1 = 0$
15. $3x^2 + 2mx - 4 = 0$
16. $ax^2 + (a-b)x + b = 0$
17. $ax^2 + 16x - 4m = 0$
18. $(a+b)x^2 - (a-b)x - a = 0$

151. The discriminant of the quadratic.—For the purpose of this section we assume that the coefficients of the quadratic are real numbers. In the solution of $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ under the radical sign "discriminates" between the different kind of roots of the equation and is called the discriminant of the quadratic.

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac = 0$, there is a double root, $-b/2a$. If $b^2 - 4ac$ is positive, the square root is a real number and there are two distinct real roots. If $b^2 - 4ac$ is negative, its square root is imaginary, and there are two distinct complex roots. If $b^2 - 4ac$ is a perfect square there are two distinct rational roots. Hence we can determine the character of the roots by examining the discriminant. Denote the discriminant by $\triangle$.

Stating this briefly, we have:

$\triangle = 0$, roots equal or coincident.
$\triangle > 0$, roots real and distinct.
  $\triangle$ a square, roots rational.
  $\triangle$ not a square, roots not rational.
$\triangle < 0$, roots complex and distinct.

1. $x^2 - 5x + 6 = 0$
   $\triangle = (-5)^2 - 4 \cdot 6 = 1$

2. $x^2 - 5x + 2 = 0$
   $\triangle = (-5)^2 - 4 \cdot 2 = 17$

3. $x^2 - 6x + 9 = 0$
   $\triangle = 36 - 4 \cdot 9 = 0$

4. $x^2 - 5x + 8 = 0$
   $\triangle = (-5)^2 - 4 \cdot 8 = -7$

EXERCISES

After forming the equations as required in these exercises, solve each equation to check the result.

1. Solve equations 1–4 at the right above, and verify that the character of the roots is determined by the discriminants.

2. In the equations on page 145 that have no literal coefficients, determine the character of the roots by finding the value of the discriminants.

3. Using the theorem on page 145, write quadratic equations whose roots are 2 and 5, 7 and -2, a and b, r and r^2, $a + b$ and $a - b$, $a + b$ and ab. Then find the discriminant of each of these equations and determine the character of the roots. Assume that a, b, r are real numbers.

4. Write the equation whose roots are $x = a + a^2$, $a - a^2$. Then solve to verify that you have the right equation.

5. Write the equation whose roots are $a + b - c$ and $a - b + c$ and verify your answer.

EXERCISES

In each of the following quadratics, *1.* find the discriminant and determine the character of the roots. 2. Solve the equation. 3. Check your solution by using §150.

1. $9x^2 - 30x + 25 = 0$
2. $3x^2 + 1 = 2x$
3. $3x + 2 = 2x^2$
4. $2x^2 + 3x - 1 = 0$
5. $1 + 2x^2 - 2x = 0$

6. $4x^2 - 3x + 1 = 0$
7. $x^2 + x + 1 = 0$
8. $s^2 + s - 1 = 0$
9. $x^2 - x + 1 = 0$
10. $x^2 - x - 1 = 0$

11. $6x^2 + 3x = 0$
12. $x^2 + 7x + 2 = 0$
13. $2x^2 - 9 = 0$
14. $x^2 - 3x - 2 = 0$
15. $-3x^2 + x + 2 = 0$

16. $-2x^2 + x - 2 = 0$
17. $x^2 = 4x - 1$
18. $x^2 + 3x + 4 = 0$
19. $2w^2 + 5w - 12 = 0$
20. $6z^2 = 7z + 3$

21. $5 = x^2 - 2x$
22. $8 = 2x^2 - 5x$
23. $x^2 - 2x + 4 = 0$
24. $m^2 + 4m + 3 = 0$
25. $n^2 - 2n + 1 = 0$

26. $2w^2 + .3w - .1 = 0$
27. $b^2 + .3b - .4 = 0$
28. $2b^2 + 4b + 3 = 0$
29. $x^2 + 2x - 4 = 0$
30. $r - 4r^2 = 5$

31. $3 - 6x = x^2$
32. $2m + 3 = m^2$
33. $5y^2 + 9y + 3 = 0$
34. $z^2 + 3z - 6 = 0$
35. $p^2 = 4 - 3p$

36. $5 = 2x^2 + 3x$
37. $7 - x^2 = 4x$
38. $3x^2 + 4x = 1$
39. $y - 5 = y^2$
40. $7x - 3x^2 = 10$

41. $2 - m^2 = 5m$
42. $2x = 4x^2 + 7$
43. $4t - t^2 - 5 = 0$
44. $0 = x^2 - 3x - 7$
45. $2s^2 - 5s - 5 = 0$

46. $4y = 5y^2 - 3$
47. $z^2 = 2z + 6$
48. $.6 = .4x + .5x^2$
49. $b + 1 = 2b^2$
50. $.5z = .6z^2 + .4$

51. $x^2 - 2ax + 4ab = b^2 + 3a^2$
52. $x^2 - abx + a^2b - ax = ab^2 - bx$
53. $2(a + 1)x^2 - (a + 1)^2x + 2(a + 1) = 4x$
54. $x^2 + 2a^2 + 3a - 2 = (3a + 1)x$
55. $(a + b)x^2 + (a - b)x + a + b = 0$

152. Factoring the general quadratic trinomial.—Heretofore we have been able to factor the trinomial $ax^2 + bx + c$ only in certain special cases. If we admit irrational and complex factors we can now find factors of this expression for all values of a, b, c. If r_1 and r_2 are roots of the equation $ax^2 + bx + c = 0$, then by the factor theorem, page 50, $x - r_1$ and $x - r_2$ are factors of $ax^2 + bx + c$. That is, $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

From the roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ it follows that the factors of $ax^2 + bx + c$ are a, $x - \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $x - \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ or a, $\dfrac{1}{2a}(2ax + b - \sqrt{b^2 - 4ac})$, and $\dfrac{1}{2a}(2ax + b + \sqrt{b^2 - 4ac})$.

That is,

$$ax^2 + bx + c = \frac{1}{4a}(2ax + b - \sqrt{b^2 - 4ac})(2ax + b + \sqrt{b^2 - 4ac})$$

If $b^2 - 4ac$, the discriminant of the quadratic, is a perfect square, these factors are rational. This is the case that we noted on page 146.

If $b^2 - 4ac = 0$, then $ax^2 + bx + c$ is a square of the type $(\sqrt{a}x + \sqrt{c})^2$. That is, $ax^2 + bx + c$ is then a perfect square in x.

EXERCISES

1. For what values of the discriminant, $b^2 - 4ac$, does the expression $ax^2 + bx + c$ have real factors? For what values are they rational and for what values are they complex?

Factor the following. Reduce complex factors to the form $P + Qi$. What special relation do you observe between these factors?

2. $6a^2 - a - 12$
3. $4a^2 - 18a - 10$
4. $3x^2 - 8x + 3$
5. $7x^2 + 4x - 8$
6. $3x^2 + 9x - 1$
7. $9x^2 - 5x + 8$
8. $3p^2 + 8p - 12$
9. $8m^2 - 3m + 1$
10. $12q^2 - 8q + 10$
11. $7r^2 - 5r + 12$
12. $10a^2 - 8a + 5$
13. $5b^2 - 3b + 7$
14. $8c^2 + 7c - 12$
15. $9d^2 - 10d + 14$
16. $3m^2 - 7m + 3$
17. $4n^2 + 5n - 12$
18. $6a^2 - 7a + 8$
19. $2p^2 - 9p + 4$
20. $7k^2 + 3k + 12$
21. $3s^2 - 10s + 8$

153. Solving higher degree equations by factoring; the cube roots of unity.—Consider an equation formed by putting a polynomial in x equal to 0. If the left member of this equation is factored, then a value of x that makes any factor equal to zero is a root of the equation. Thus $x = 1$, $x = -2$, $x = 3$, are roots of the equation $(x-1)(x+2)(x-3) = 0$. Again, $x = a$ is a root of the equation $(x - a)$(any polynomial in x) $= 0$, since $x = a$ makes the factor $x - a$ equal to zero. In this case we may know nothing about values of x that make the polynomial in the second parenthesis equal to zero, or may not be able to find any such value.

In the equation $x^3 = 1$ when put in the form $x^3 - 1 = 0$, the left member may be factored into the form $(x-1)(x^2 + x + 1)$. Then the root $x = 1$ is at once apparent. To find values that make the second factor equal to zero we solve the equation $x^2 + x + 1 = 0$. Clearly the values of x thus found will make the whole left member equal to zero and hence satisfy the equation $x^3 - 1 = 0$, or $x^3 = 1$. That is, these results are cube roots of unity.

$$x^3 = 1$$
$$x^3 - 1 = 0$$
$$(x-1)(x^2+x+1) = 0$$
$$x = 1$$
$$x = \frac{-1 \pm \sqrt{3}i}{2}$$
$$\sqrt[3]{1} = 1, \quad w, w^2$$

These cube roots of unity have the interesting property that the square of either of them is equal to the other. That is,

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-1-\sqrt{3}i}{2} \quad \text{and} \quad \left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \frac{-1+\sqrt{3}i}{2}$$

The complex cube roots of unity are often denoted by ω and ω^2.

EXERCISES

1. Square the two complex roots of unity and thus prove that the last statement above involves no contradiction.

2. Show that the product of the two complex cube roots of unity is equal to 1.

3. Cube $\dfrac{-1+\sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$. The results should be unity.

4. By solving $x^3 + 1 = (x+1)(x^2 - x + 1)$ find the three cube roots of -1.

5. Solve $x^4 - 1 = 0$ and find the four fourth roots of unity. Note that $x^4 - 1 = (x^2 - 1)(x^2 + 1)$.

154. The cube roots of any real number; the fourth roots of a positive number.—If $\sqrt[3]{a}$ is the real cube root of a real number a, $\sqrt[3]{a}\left(\frac{1 \pm \sqrt{3}i}{2}\right)$ are cube roots of a since the cube of the expression in parenthesis is 1.

Cube roots of a: $\sqrt[3]{a}$, $\sqrt[3]{a}\left(\frac{1 \pm \sqrt{3}i}{2}\right)$
Fourth roots of a: $\sqrt[4]{a}$, $-\sqrt[4]{a}$, $\sqrt[4]{a}i$, $-\sqrt[4]{a}i$

If $\sqrt[4]{a}$ is the real fourth root of a positive number a, then the expressions given at the right above are obviously fourth roots of a.

155. Equations reducible to quadratics.—Below are given examples of types of equations that can be solved as quadratics.

$$ax^4 + bx^2 + c = 0$$
$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \pm\sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example 1. Solve $x^4 - 7x^2 + 10 = 0$. If we substitute $x^2 = y$, then $y^2 - 7y + 10 = 0$, and $y = 5, 2$. Hence $x = \pm\sqrt{5}, \pm\sqrt{2}$. We may avoid substituting by regarding the given equation as a quadratic in x^2 and hence obtain at once $x^2 = 5, 2$. The solution above at the right shows the general case $ax^4 + bx^2 + c = 0$.

Example 2. Solve $x^6 - 9x^3 + 8 = 0$. We have $x^3 = 1, 8$. Hence to find the complete solution we need to find the three cube roots of 1 and 8.

$$x^6 - 9x^3 + 8 = 0$$
$$x^3 = 1,\ 8$$
$$x = 1,\ \frac{-1 \pm \sqrt{3}i}{2}$$
$$x = 2,\ 2\left(\frac{-1 \pm \sqrt{3}i}{2}\right) = -1 \pm \sqrt{3}i$$

EXERCISES

Solve the following equations. Find all the roots. Check by going over the work with care.

1. $x^4 + 3x^2 + 2 = 0$
2. $x^4 - 5x^2 + 6 = 0$
3. $x^4 + 5x^2 + 6 = 0$
4. $x^4 - 7x^2 + 12 = 0$
5. $x^4 + 9x^2 + 8 = 0$
6. $x^8 - 3x^4 + 2 = 0$
7. $x^8 - 10x^4 + 9 = 0$
8. $x^4 - 2x^2 + 1 = 0$
9. $x^8 - 2x^4 + 1 = 0$
10. $x^6 - 29x^3 + 54 = 0$
11. $x^6 + 29x^3 + 54 = 0$
12. $x^6 - 7x^3 - 8 = 0$
13. $x^6 + 7x^3 - 8 = 0$
14. $3x^4 + 7x^2 - 6 = 0$
15. $6x^4 + 5x^2 - 4 = 0$
16. $5x^4 + x^2 - 18 = 0$
17. $4x^6 + x^3 - 3 = 0$
18. $6x^6 - 17x^3 + 10 = 0$

Example 3. Solve $x^2 - 5x + 4 - 3\sqrt{x^2 - 5x + 4} + 2 = 0$. If we let $y = \sqrt{x^2 - 5x + 4}$, then the equation reduces to $y^2 - 3y + 2 = 0$. Then $y = 1, 2$ and we need to solve each of the equations $\sqrt{x^2 - 5x + 4} = 1$ and $\sqrt{x^2 - 5x + 4} = 2$.

Squaring both members in each equation, we have the solution shown at right.

$$\begin{aligned} \sqrt{x^2 - 5x + 4} &= 1 \\ x^2 - 5x + 3 &= 0 \\ x &= \frac{5 \pm \sqrt{13}}{2} \\ \sqrt{x^2 - 5x + 4} &= 2 \\ x^2 - 5x &= 0 \\ x &= 0, 5 \end{aligned}$$

Example 4. Solve $x^2 - 7x - 6\sqrt{x^2 - 7x - 5} - 21 = 0$. To solve this equation it is necessary to add and subtract 5 to make the expression outside the radical sign the same as that under this sign. Then we have

$$x^2 - 7x - 5 - 6\sqrt{x^2 - 7x - 5} - 16 = 0$$

Solving this quadratic in $\sqrt{x^2 - 7x - 5}$ we have $\sqrt{x^2 - 7x - 5} = 8, -2$ and hence $x^2 - 7x - 5 = 64$ and $x^2 - 7x - 5 = 4$.

These give $x = \dfrac{7 \pm \sqrt{325}}{2}$ and $x = \dfrac{7 \pm \sqrt{85}}{2}$.

This solution is possible only when the coefficients of x^2 and x are the same in the expressions under the radical and outside the radical. In the above case a number can be added or subtracted that will reduce the equation to the form $ax^2 + bx + c + p\sqrt{ax^2 + bx + c} + q = 0$.

Checking solutions by substitution in Examples 3 and 4 is a little complicated. Going over the work with care to insure accuracy is much simpler.

The equation in Example 1 is of the fourth degree. Using the possible combinations of the signs in the answer, we have four values of x. In Example 2, the equation is of the sixth degree, and we find six values of x. In Examples 3 and 4 we have four values of x, and these equations are of the fourth degree when the radicals are removed.

EXERCISES

Solve the following.

1. $x^2 - 3x + 2 - 5\sqrt{x^2 - 3x + 2} + 6 = 0$
2. $x^2 + 5x - 5 - 9\sqrt{x^2 + 5x - 5} + 14 = 0$
3. $x^2 - 2x + 3 - 6\sqrt{x^2 - 2x + 3} + 5 = 0$
4. $x^2 + 4x + 2 - 4\sqrt{x^2 + 4x + 5} + 6 = 0$
5. $x^2 - 9x + 3 + 6\sqrt{x^2 - 9x + 4} - 6 = 0$
6. $x^2 - 5x + 2 - 4\sqrt{x^2 - 5x + 7} + 8 = 0$
7. $x^2 - 8x + 3 + 3\sqrt{x^2 - 8x - 3} - 9 = 0$
8. $x^2 + 5x - 7 - 8\sqrt{x^2 + 5x + 2} - 4 = 0$

PROBLEMS

The two groups of problems, A and B, are of about equal difficulty. Either group may be used.

Group A

1. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips. Also solve this problem if it is required to multiply the area of the original rectangle by k.

2. A city block is 400 by 480 feet when measured to the outer edge of the sidewalk. At 4 cents per square foot it costs \$416.64 to lay a sidewalk around the block. Find the width of the walk.

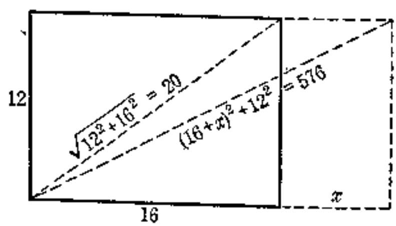
3. The sides of a right triangle are 6 inches and 8 inches respectively. How many inches must be added to each side so as to increase the hypotenuse 10 inches?

4. A picture is 15 inches by 20 inches. How wide a frame must be added to increase the diagonal 3 inches?

5. *A* starts north from a certain place going 4 miles per hour and *B* starts east from the same place at the same time going 3 miles per hour. In how many hours will they be 16 miles apart, the earth's surface being considered as a plane?

6. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

7. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches?



8. How much must the *width* of the rectangle in problem 7 be extended so as to increase the diagonal by 4?

9. A certain university campus is 100 rods long and 80 rods wide. There are two driveways running through the center of the campus at right angles to each other and parallel to the sides. What is the width of these driveways if their combined area is 356 square rods?

10. A room is 5 feet longer than it is wide and the distance between two opposite corners is 25 feet. Find the length and width of the room.

11. A vacant corner lot has a 50-foot frontage on one street. What is the frontage on the other street if the distance between opposite corners along the diagonal is 10 feet less than twice this frontage?

12. The sum of the squares of two consecutive integers is 13,945. Find the numbers.

13. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

14. Find the length of a side of an equilateral triangle whose altitude is 2 feet shorter than a side.

15. A motorboat takes 2 hours to travel 8 miles downstream and 4 miles back on a river which flows at the rate of 2 miles per hour. Find the rate at which the motorboat would travel in still water.

16. The sum of the reciprocals of two consecutive numbers is 15/56; find the numbers.

17. If a train traveled 5 miles an hour faster, it would take one hour less to travel 210 miles; what time does it take?

18. I buy a number of footballs for $100; had they cost a dollar apiece less, I should have had five more for the money; find the cost of each.

Group B

1. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

2. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must he cut to make 12 acres?

3. A rectangular lot is 16 by 12 rods. How wide a strip must be added to one end and one side to obtain a rectangular lot whose diagonal is 1 rod greater?

4. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) will be increased by 100 feet. What will be the dimensions of the new field?

5. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one-tenth of the whole piece?

6. A trunk 30 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally at the bottom. How much must the length of the trunk be increased if it is to accommodate a gun 4 inches longer than the umbrella?

7. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park, and a drive through the length of the park parallel to the sides. What is the width of the walk if the drive is three times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

8. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of its hypotenuse and of the remaining side.

9. The product of two consecutive integers is 4422. Find the numbers.

10. A square piece of tin is made into an open box containing 864 cubic inches by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.

11. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide.

12. A train approaching Chicago from the south at the rate of 50 miles per hour is 75 miles away when a train starts west from Chicago at the rate of 25 miles per hour. How long after the second train starts will they be 50 miles apart measured diagonally across the country?

13. The side of one square is 3 feet longer than the side of a second square and the area of the first square is twice the area of the second. Find the area of each square.

14. After traveling for 50 miles at a certain speed, a motorist increases his speed by 10 miles per hour and travels 100 miles farther. If he took 3 hours to cover the 150 miles, find his speed during the first 50 miles.

15. Find a number which when increased by 17 is equal to 60 times the reciprocal of the number.

16. The sum of a number and its square is nine times the next higher number; find it.

17. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way it would take 128 tiles; find the length of each tile.

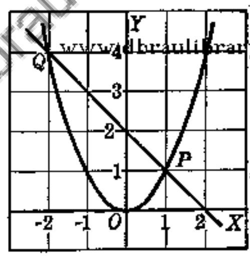
CHAPTER 12:

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

In finding the points in which the parabola $y = ax^2 + bx + c$ meets the x-axis we have already solved simultaneously the quadratic equation $y = ax^2 + bx + c$ and the first degree equation $y = 0$. We shall now make a more general study of two equations in two variables, one being of the second degree and one of the first degree, or both being of the second degree.

$$\begin{aligned} y &= ax^2 + bx + c \\ y &= 0 \end{aligned}$$

156. The parabola $y = x^2$ *and the straight line.*—Consider the parabola $y = x^2$ and the straight line $x + y = 2$. Solving these equations simultaneously will give the coordinates of the points in which the line intersects the parabola.



The solution is as follows.

Step 1. Solve (2) for y and substitute in (1). This gives equation (4).

Step 2. Solve (4) obtaining $x = 1, -2$.

Step 3. Substitute $x = 1$ and $x = -2$ in (2), obtaining $y = 1$ and $y = 4$.

The solutions are shown at the right. Note that the pair $x = 1$, $y = 1$ constitutes one solution (intersection point), while $x = -2$, $y = 4$ constitutes the other. Check by finding whether these are the coordinates of P and Q in the figure.

$$\left.\begin{aligned} y &= x^2 \\ x + y &= 2 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

$$y = 2 - x \qquad (3)$$

$$2 - x = x^2 \qquad (4)$$

$$x^2 + x - 2 = 0 \qquad (5)$$

$$x = 1, \quad -2$$

$$\begin{aligned} x &= 1 \qquad & x &= -2 \\ y &= 1 \qquad & y &= 4 \end{aligned}$$

EXERCISES

1. Find the coordinates of the points in which the lines $x + y = 4$ and $x - 3y = 6$ intersect the parabola $y = x^2$. Construct the lines and the parabola to see whether you have found the coordinates of the intersection points.

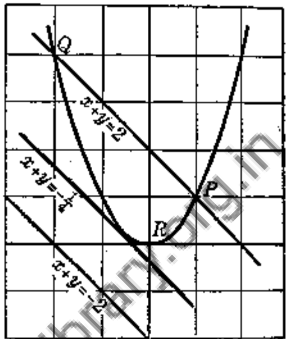
157. Imaginary solutions; tangents.—The figure shows the parabola $y = x^2$ and the lines $x + y = 2$, $x + y = -\frac{1}{4}$, and $x + y = -2$.

$$\left.\begin{aligned} y &= x^2 \\ x + y &= 2 \end{aligned}\right\} \qquad \left.\begin{aligned} y &= x^2 \\ x + y &= -2 \end{aligned}\right\}$$

$$\begin{matrix} x = 1 & x = -2 \\ y = 1 & y = 4 \end{matrix}\Bigg\} \qquad \left.\begin{aligned} x &= \frac{-1 \pm \sqrt{7}i}{2} \\ y &= \frac{-3 \mp \sqrt{3}i}{2} \end{aligned}\right\}$$

$$\left.\begin{aligned} y &= x^2 \\ x + y &= -\tfrac{1}{4} \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= -\tfrac{1}{2} \\ y &= \tfrac{1}{4} \end{aligned}\right\}$$



Solving $y = x^2$ and $x + y = 2$ simultaneously gives the coordinates of the distinct points P and Q in the figure (see page 155).

Solving $y = x^2$, $x + y = -\frac{1}{4}$ by substituting $y = -x - \frac{1}{4}$ in $y = x^2$ gives the double root $x = \dfrac{-1 \pm \sqrt{0}}{2} = -\dfrac{1}{2}, -\dfrac{1}{2}$. This shows that this line meets the curve in only one point (a double point) and hence is tangent to it. The solution of $y = x^2$, $x + y = -2$ is not real, and hence the line does not meet the curve.

We can now find a meaning for the statement that a tangent to a curve meets it in two "coincident" points. In the equation $x + y = a$ let a begin with some number such as 2, and take a series of values approaching $-\frac{1}{4}$. Then the points P and Q will move along the curve toward R and as a becomes $-\frac{1}{4}$ these points will fall together or coincide at R. This can be shown graphically by placing a ruler on the parabola and moving it until it becomes tangent to the curve.

$$\begin{aligned} y &= x^2 \\ x + y &= a \\ y &= a - x \\ x^2 + x - a &= 0 \\ x &= \frac{-1 \pm \sqrt{1 + 4a}}{2} \end{aligned}$$

Solving $y = x^2$ and $x + y = a$ gives the result shown at the right. It is evident that for $a = -\frac{1}{4}$ this gives $\dfrac{-1 \pm \sqrt{0}}{2} = -\dfrac{1}{2}, -\dfrac{1}{2}$ for x. If $a > -\frac{1}{4}$, the discriminant is positive and we have two distinct real values, while for $a < -\frac{1}{4}$ the values are complex numbers.

Example 1. For what values of b will the parabola $y = x^2 + bx + 4$ meet the x-axis in two coincident points? In two distinct points? in no point?

$$
\begin{aligned}
&y = x^2 + bx + 4\\
&y = 0\\
&x^2 + bx + 4 = 0\\
&x = \frac{-b \pm \sqrt{b^2 - 16}}{2}
\end{aligned}
$$

SOLUTION: Solving $y = 0$ and $y = x^2 + bx + 4$ simultaneously gives $x = \dfrac{-b \pm \sqrt{b^2 - 16}}{2}$. If $b = 4$, the parabola meets the x-axis in one point (two coincident points), $x = -4/2 = -2$. If $b > 4$, the parabola meets the axis in two distinct points; and if $b < 4$, it fails to meet the axis. Give reasons for these statements.

Example 2. For what values of a will the line $ax + y = 4$ be tangent to the parabola $y = x^2 - 4x + 5$?

$$
\begin{aligned}
&y = x^2 - 4x + 5 \qquad (1)\\
&ax + y = 4 \qquad (2)\\
&x = \frac{-a + 4 \pm \sqrt{a^2 - 8a + 12}}{2}
\end{aligned}
$$

In the equations at the right solve (2) for y and substitute in (1). Solve the resulting equation for x, obtaining the values shown at the right.

In order that the line shall be tangent to the curve the expression under the radical must be zero. Solving this equation gives $a = 6, 2$.

$$
\begin{aligned}
&a^2 - 8a + 12 = 0\\
&a = \frac{8 \pm \sqrt{64 - 48}}{2}\\
&\quad = \frac{8 \pm 4}{2}, = 6,\ 2
\end{aligned}
$$

Construct the parabola and the two lines $6x + y = 6$ and $2x + 6 = 6$. They should both be tangent to the curve.

EXERCISES

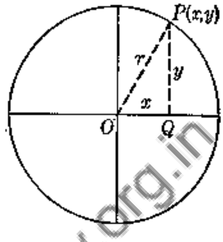
In each of the following find the intersection points of the parabola and the line.

1. $y = x^2 - 3x + 2$, $x + y = 4$
2. $y = x^2 + 3x - 6$, $2x + y = 3$
3. $y = x^2 + 6x - 10$, $x - 3y + 4 = 0$

In each of the following find the value of c that makes the line tangent to the curve. For what values of c will the line meet the curve in two distinct points? in no point?

4. $y = x^2 + 3x + c$, $x - y = 2$
5. $y = x^2 - 7x - c$, $2x - y = 6$
6. $y = -x^2 + 4x + c$, $2x - 3y = 6$
7. $y = x^2 - 4x + 2$, $x + 2y = c$
8. $y = x^2 - 6x + 8$, $x + cy = 5$
9. $y = x^2 + 6x - 6$, $cx + 3y - 9 = 0$
10. $y = 6x^2 + 3x - 2$, $x + y = -c$
11. $y = x^2 + cx - 2$, $x - y = 3$
12. $y = 3x^2 - 2x - c$, $3x + y = 2$

158. The equation of the circle.—Consider a circle with radius r and center at the origin. Let $P(x,y)$ be any point on this circle. Then x and y are two sides of the right triangle OQP in the figure, and r is the hypotenuse. From the right triangle theorem, $x^2 + y^2 = r^2$. This holds for every point $P(x,y)$ on the circle and for no other point in the plane. Hence $x^2 + y^2 = r^2$ is the equation of the circle.



EXERCISES

1. Find the points in which $x + y = 3$ meets the circle $x^2 + y^2 = 16$.
SUGGESTION: Explain the steps shown at the right. Then solve equation (4), finding $x = \frac{3 \pm \sqrt{23}}{2}$. Substitute these values in (2) and find the corresponding values of y. Construct the circle and the line and then verify your solution. Use $\sqrt{23} = 4.8$.

$$\left.\begin{array}{l} x^2 + y^2 = 16 \\ x + y = 3 \end{array}\right\} \quad \begin{array}{r} (1) \\ (2) \end{array}$$
$$x^2 + (3 - x)^2 = 16 \quad (3)$$
$$2x^2 - 6x - 7 = 0 \quad (4)$$

2. For what values of a will the line $x - y = a$ meet the circle $x^2 + y^2 = 25$ in two distinct points? For what value (values?) of a will the line be tangent to the circle? For what values of a will the line fail to meet the circle? Your answer to the second question should be $a = \pm 5\sqrt{2}$.

Note that for different values of a the lines $x - y = a$ are parallel. Construct $x - y = 1$ and $x - y = 4$ to verify this statement. Why should there be two values of a for which the line $x + y = a$ is tangent to the circle? If $a < -5\sqrt{2}$ or $a < 5\sqrt{2}$, what is the character of the solution? What is the character of the solution if $-5\sqrt{2} < a < 5\sqrt{2}$?

3. For what values of a will the line $x + y = a$ be tangent to the circle $x^2 + y^2 = 16$? For what values of a will the line meet the circle in two distinct points? For what values of a will this line fail to meet the circle?

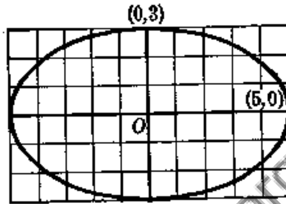
4. For what values of r will the line $x + y = 6$ be tangent to the circle $x^2 + y^2 = r^2$? For what values of r will the line meet the circle in two points?

5. Find the points in which the line $y = mx + 4$ meets the circle $x^2 + y^2 = 25$. Study your solution to answer the following questions. (a) Is there any value of m for which the line fails to meet the circle? (b) Is there any value of m for which the line is tangent to the circle? Note that m must be a real number.

6. Find a relation between m and r for which the line $y = mx + 4$ is tangent to the circle $x^2 + y^2 = r^2$.

7. Find a relation between a and r for which the line $x + y = a$ is tangent to the circle $x^2 + y^2 = r^2$.

159. The equation of the ellipse.—We know in a general way that an ellipse is a smooth oval figure. The ellipse is defined mathematically as the locus of an equation of the type $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We shall study the equation $\frac{x^2}{25}+\frac{y^2}{9}=1$. Solving this equation for y gives $y=\pm\frac{3}{5}\sqrt{25-x^2}$. Verify this solution. Then check the following pairs of values to see whether they satisfy the equation. Substitute each value of x and find the corresponding value of y.



x	0	0	5	-5	4	4	3	3	-4	-4	-3	-3
y	3	-3	0	0	$\frac{9}{5}$	$-\frac{9}{5}$	$\frac{12}{5}$	$-\frac{12}{5}$	$\frac{9}{5}$	$-\frac{9}{5}$	$\frac{12}{5}$	$-\frac{12}{5}$

Plot these points and draw a smooth curve through them. The resulting curve is an ellipse.

EXERCISES

1. If in $\frac{x^2}{25}+\frac{y^2}{9}=1$ you substitute a value of x that is greater than 5 or less than -5, what is the character of the corresponding value of y?
If $y=4$ or $y=-4$, what is the character of the corresponding values of x?
SUGGESTION: Use the value $y=\pm\frac{3}{5}\sqrt{25-x^2}$ obtained above.

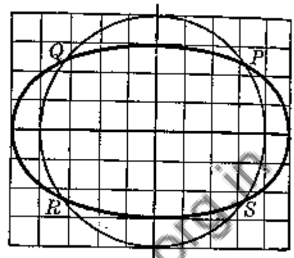
2. For the set of all points on this curve what are the limits of the abscissas? of the ordinates?

3. Find the coordinates of the points in which $x+y=3$ meets this ellipse. Verify your solution by constructing the line and the ellipse.

4. For what values of a is the line $x+y=a$ tangent to this ellipse? For what values of a does this line meet the ellipse in two distinct points? in no point?

5. Answer the questions in example 4 for the line $x-y=a$. Remember that a must have real values.

160. The circle and the ellipse.—The figure shows the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. To solve these equations regard them as first degree equations in x^2 and y^2.



Then $x^2 = \frac{175}{16}$, $\quad y^2 = \frac{81}{16}$

Hence $x = \pm \frac{5}{4}\sqrt{7}$, $\quad y = \pm \frac{9}{4}$

Then what are the coordinates of P?

Clearly these are both positive and we have $P\left(\frac{5}{4}\sqrt{7}, \frac{9}{4}\right)$. From the figure it is then obvious that we get the four solutions:

$P\left(\frac{5}{4}\sqrt{7}, \frac{9}{4}\right)$, $\quad Q\left(-\frac{5}{4}\sqrt{7}, \frac{9}{4}\right)$, $\quad R\left(-\frac{5}{4}\sqrt{7}, -\frac{9}{4}\right)$ $\quad S\left(\frac{5}{4}\sqrt{7}, -\frac{9}{4}\right)$

The pairing of the values of x and y involved in finding these four solutions is easily determined from the original equations without reference to the figure. If $x = x_1$ and $y = y_1$ satisfy both equations, then since $x_1^2 = (-x_1)^2$, $y_1^2 = (-y_1)^2$, it is obvious that $(\pm x_1)^2 + (\pm y_1)^2 = 16$ and $\frac{(\pm x_1)^2}{25} + \frac{(\pm y_1)^2}{9} = 1$. Hence any of the pairs (x_1, y_1), $(-x_1, y_1)$, $(-x_1, -y_1)$, $(x_1, -y_1)$ will satisfy both equations.

$$\frac{x_1^2}{25} + \frac{y_1^2}{9} = 1 \quad (1)$$
$$x_1^2 + y_1^2 = 16 \quad (2)$$

EXERCISES

1. Copy the ellipse on page 159 and on the same axes construct the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 + y^2 = 25$, and $x^2 + y^2 = 36$. Use paper ruled in large units to make figure clear.

2. Solve simultaneously with $\frac{x^2}{25} + \frac{y^2}{9} = 1$ each of the equations $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 + y^2 = 25$, $x^2 + y^2 = 36$. Explain the character of each solution.

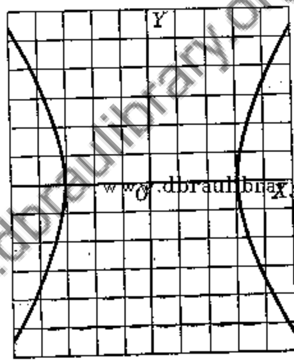
3. Solve simultaneously $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and $x^2 + y^2 = r^2$. For what values of r will these curves meet in four points? for what values of r will they meet in two points? for what values of r will they fail to meet?

161. The equation of the hyperbola.—Consider the equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Check the steps at the right used in finding y in terms of x. Check to see if the following pairs of values of x and y satisfy the given equation.

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$
$$y^2 = \frac{16}{9}(x^2 - 9)$$
$$y = \pm\frac{4}{3}\sqrt{x^2 - 9}$$

x	3	-3	5	-5	4	-4
y	0	0	$\pm\frac{16}{3}$	$\pm\frac{16}{3}$	$\pm\frac{4}{3}\sqrt{7}$	$\pm\frac{4}{3}\sqrt{7}$

Construct these points and draw a smooth curve through them. Use $\sqrt{7} = 2.6$. Note that there is no point on the curve for which x is between -3 and $+3$. The curve is called a hyperbola. Note that it consists of two separate branches.



EXERCISES

1. Find the points in which $y - x = 4$ meets this curve. Construct the line on the same axes as your hyperbola and verify your solution.

2. For what values of a will $y - 2x = a$ be tangent to this hyperbola? For what values will the line meet the hyperbola in two distinct points, and for what values will the line fail to meet it?

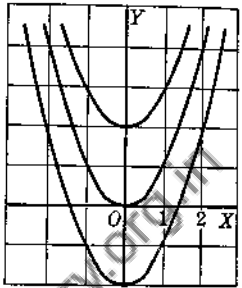
3. Solve each of the equations $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 + y^2 = 16$ simultaneously with the above equation of the hyperbola. Use figures to explain why each solution is as you find it.

4. Solve simultaneously $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $\frac{x^2}{9} - \frac{y^2}{16} = 1$, and also $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $\frac{x^2}{25} - \frac{y^2}{9} = 1$. Explain why the solution is as you find it.

SUGGESTION: Solve the equation for x^2 and y^2 and then take square roots. Arrange the values of x and y in four pairs to give the coordinates of the four intersection points. Compare the intersections of the circle and the ellipse studied on page 160.

5. Is the line $x + y = c$ tangent to the curve $\frac{x^2}{9^2} - \frac{y^2}{16} = 1$ for any real value of c?

162. Intersections of circle and parabola.—For different values of a, $y = x^2 + a$ represents a set of parabolas like those shown at the right. Finding the intersection points of such parabolas and circles $x^2 + y^2 = r^2$ for different values of r illustrates some of the interesting properties of quadratic equations.



PROBLEM. Solve simultaneously $x^2 + y^2 = r^2$ and $y = x^2 + a$.

SOLUTION: From $y = x^2 + a$, $x^2 = y - a$. Substituting in $x^2 + y^2 = r^2$, $y^2 + y - a - r^2 = 0$ and hence,

$$y = \frac{-1 \pm \sqrt{1 + 4a + 4r^2}}{2}$$

Substituting these values of y in $x^2 = y - a$,

$$x = \pm \sqrt{\frac{-1 - 2a \pm \sqrt{1 + 4a + 4r^2}}{2}}$$

Then we have the four solutions:

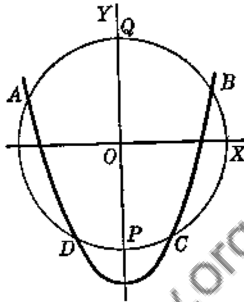
$$A \begin{cases} x = \sqrt{\dfrac{-1-2a+\sqrt{1+4a+4r^2}}{2}} \\ y = \dfrac{-1+\sqrt{1+4a+4r^2}}{2} \end{cases} \qquad B \begin{cases} x = -\sqrt{\dfrac{-1-2a+\sqrt{1+4a+4r^2}}{2}} \\ y = \dfrac{-1+\sqrt{1+4a+4r^2}}{2} \end{cases}$$

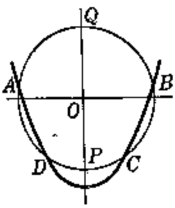
$$C \begin{cases} x = \sqrt{\dfrac{-1-2a-\sqrt{1+4a+4r^2}}{2}} \\ y = \dfrac{-1-\sqrt{1+4a+4r^2}}{2} \end{cases} \qquad D \begin{cases} x = -\sqrt{\dfrac{-1-2a-\sqrt{1+4a+4r^2}}{2}} \\ y = \dfrac{-1-\sqrt{1+4a+4r^2}}{2} \end{cases}$$

$x^2 + y^2 = r^2$	(1)
$y = x^2 + a$	(2)

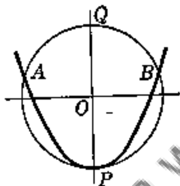
The pairing of the values of x and y is now a little different from that on page 160. If x_1, y_1 satisfy both equations, then clearly $-x_1, y_1$ will also satisfy them, while $-x_1$, $-y_1$ will not satisfy them. If we use the $+$ sign before the radical in the value of y and substitute, we get the solutions A and B above, and if we use the $-$ sign before this radical we get the solutions C and D.

If the parabola meets the circle in four distinct points, then the solutions A, B, C, D represent the points A, B, C, D in the figure. In this case all the solutions are real. Clearly, raising the parabola will make the points C, D approach P, where these will coincide. This gives two distinct points A and B, the one "double" point P in which the curves meet. Raising the parabola further will leave only the two intersection points A and B. Hence the solutions C and D will be imaginary. Continued raising of the parabola will bring its lowest point up to Q and then above Q where all the solutions will be imaginary.

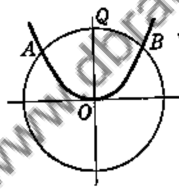




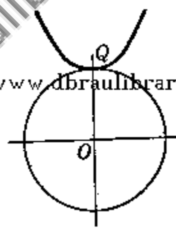
1. $x^2 + y^2 = 16$
$y = x^2 - 8$



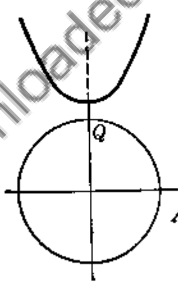
2. $x^2 + y^2 = 16$
$y = x^2 - 4$



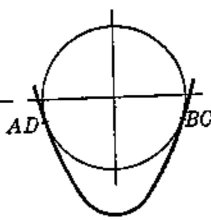
3. $x^2 + y^2 = 16$
$y = x^2$



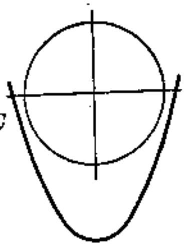
4. $x^2 + y^2 = 16$
$y = x^2 + 4$



5. $x^2 + y^2 = 16$
$y = x^2 + 5$



6. $x^2 + y^2 = 16$
$y = x^2 - 16\frac{1}{4}$



7. $x^2 + y^2 = 16$
$y = x^2 - 20$

EXERCISES

1. Substituting $a = -8$, $r = 4$ in the general solution on page 162, find the coordinates of A, B, C, D in 1 page 163. Note that $\sqrt{1 + 4a + 4r^2} = \sqrt{1 - 32 + 64} = \sqrt{33}$ occurs in all values of x and y. Find this correct to two decimals. Find all final results correct to one decimal. Note also that in A and B the values of x are simply of opposite signs, while the values of y are the same for these two points. This is also true for the points C and D.

2. Solve in succession the equations in 2, 3, . . . , 7 on page 163. Study the corresponding figures and explain the meaning of each solution.

Note that the values of one of the unknowns (x or y) may be real and the other a complex number. Such a pair of values constitute an imaginary solution.

163. ***Special examples involving quadratics.***—Many cases of simultaneous equations may be reduced to the solution of a quadratic by using more or less obvious artifices. In the following series of such Examples suggestions are given that should enable you to solve them without further help. In each case the equations to be solved are given at the right. Suggestions are given at the left. Solve each set completely.

Example 1. Factor $x^3 + y^3$ and substitute $x + y = 3$. Solve the resulting equation simultaneously with (2). Also solve by substituting $y = 3 - x$ in $x^3 + y^3 = 9$.

1. $x^3 + y^3 = 9$ (1), $x + y = 3$ (2)

Example 2. Factor $x^3 - y^3$ and substitute $x - y = 2$ as in Example 1. Also solve by substituting from (2) in (1).

2. $x^3 - y^3 = 19$ (1), $x - y = 1$ (2)

Example 3. Add $2xy = 16$ and subtract $2xy = 16$ from (1) and take square roots. Solve the resulting equations. Also solve by substituting from (2) in (1).

3. $x^2 + y^2 = 20$ (1), $xy = 8$ (2)

Example 4. Substitute $1/x = a$, $1/y = b$, and solve $a + b = 5$, $a^2 + b^2 = 13$.

4. $\frac{1}{x} + \frac{1}{y} = 5$ (1), $\frac{1}{x^2} + \frac{1}{y^2} = 13$ (2)

Example 5. Divide (1) by y^2, substitute a for x/y, and solve $a^2 - 3a + 2 = 0$. Find $x = y$, $x = 2y$. Substitute in (2) and solve for y.

5. $x^2 - 3xy + 2y^2 = 0$ (1), $3x^2 + 5xy - y^2 = 7$ (2)

Example 6. Solve (1) for x^2y^2 and then for xy. (If necessary substitute $a = x^2y^2$.) Then solve with (2) as in Example 3.

6.
$$x^4y^4 + x^2y^2 = 90 \quad (1)$$
$$x^2 + y^2 = 10 \quad (2)$$

7.
$$\frac{16}{x^2} + \frac{81}{25y^2} = 1 \quad (1)$$
$$\frac{9}{x^2} + \frac{144}{25y^2} = 1 \quad (2)$$

Example 7. Let $1/x^2 = a$, $1/y^2 = b$. Then solve $16a + \frac{81}{25}b = 1$, $9a + \frac{144}{25}b = 1$. These equations must be solved if we are to find a^2 and b^2 in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that will make the ellipse pass through the points (4, 9/5) and (3, 12/5).

8.
$$5x^2 - 3xy + 4y^2 = 6 \quad (1)$$
$$3x^2 - 4xy + 3y^2 = 2 \quad (2)$$

Example 8. The left members of these equations are homogeneous in x and y. In such cases substitute $y = vx$ and solve for x^2, obtaining,

$$x^2 = \frac{6}{5 - 3v + 4v^2}, \; x^2 = \frac{2}{3 - 4v + 3v^2}$$

Hence $$\frac{6}{5 - 3v + 4v^2} = \frac{2}{3 - 4v + 3v^2}, \text{ and } v = 1, 4/5.$$

Substitute $y = x$ and also $y = 4/5\ x$ in (1) or (2) and solve for x. Then using $y = x$ and $y = 4/5\ x$ find the corresponding values of y.

9.
$$x^3 + y^3 = a\,(x+y) \quad (1)$$
$$x\,y = b \quad (2)$$

Example 9. Factor $x^3 + y^3$ and divide out $x+y$, obtaining $x^2 - xy + y^2 = a$. Solve simultaneously with (2) using a plan similar to that in Example 3. For what relation between a and b are the solutions real? Be careful to consider both radicals in your solution.

10.
$$x^2y + xy^2 = -16 \quad (1)$$
$$x + y = 2 \quad (2)$$

Example 10. Factor (1) and substitute $x + y = 2$. Then solve the result simultaneously with (2).

11.
$$x^2 + y^2 + x + y = 14 \quad (1)$$
$$xy = 3 \quad (2)$$

Example 11. Multiply (2) by 2 and add to (1). Then $(x + y)^2 + (x + y) = 20$. Solve for $x + y$ and solve simultaneously with (2). This set of equations should have four solutions.

12.
$$x^4 + y^4 = 82 \quad (1)$$
$$x + y = 4 \quad (2)$$

Example 12. Raise (2) to the 4th power and subtract (1), obtaining $4x^3y + 6x^2y^2 + 4xy^3 = 174$, or

$$2xy\,(2x^2 + 3xy + 2y^2) = 174 \quad (3)$$

Square (2). $x^2 + 2xy + y^2 = 16$, or $2x^2 + 4xy + 2y^2 = 32$. Hence $2x^2 + 3xy + 2y^2 = 32 - xy$. Substituting in (3), $2xy\,(32 - xy) = 174$. Solve for xy and then solve simultaneously with (2).

EXERCISES

Solve the following pairs of equations simultaneously.

1. $r^2 + rs + s^2 = 63$
   $r - s = 3$

2. $x^2 + y^2 = a$
   $xy = b$

3. $x^3 + y^3 = 91$
   $x + y = 7$

4. $x^2 + y^2 = a$
   $x^2 - y^2 = b$

5. $x^2 - 3xy = 0$
   $5x^2 + 3y^2 = 9$

6. $3x - 2y = 6$
   $3x^2 - 2xy + 4y^2 = 12$

7. $3x^2 + 2xy = 16$
   $4x^2 - 3xy = 10$

8. $a^2 + ab + b^2 = 7$
   $a^2 - ab + b^2 = 19$

9. $a + b + ab = 11$
   $(a + b)^2 + a^2b^2 = 61$

10. $ax - by = 0$
    $x^2 + y^2 = c$

11. $\frac{1}{x^3} + \frac{1}{y^3} = 19$

    $\frac{1}{x} + \frac{1}{y} = 1$

12. $4a^2 - 2ab = b^2 - 16$
    $5a^2 = 7ab - 36$

13. $3x^2 - 9y^2 = 12$
    $2x - 3y = 14$

14. $x^2 + xy + y^2 = a$
    $x^2 + y^2 = b$

15. $x^2 + y^2 + x + y = 18$
    $xy = 6$

16. $x^2 - 5xy + y^2 = -2$
    $x^2 + 7xy + y^2 = 22$

17. $a^2 + 6ab + b^2 = 124$
    $a + b = 8$

18. $a^2 - 3ab + 2b^2 = 0$
    $2a^2 + ab - b^2 = 9$

19. $x^2 + y^2 + 2x + 2y = 27$
    $xy = -12$

20. $x^2 + y^2 - 5x - 5y = -4$
    $xy = 5$

21. $(7 + x)(6 + y) = 80$
    $x + y = 5$

22. $\left(\frac{9}{x}\right)^2 = \left(\frac{25}{y}\right)^2 - 16$

    $\frac{9}{x^2} = \frac{25}{y^2}$

23. $2x^2 - 5xy + 3x - 2y = 22$
    $5xy + 7x - 8y - 2x^2 = 8$

24. $(x - 4)^2 + (y + 4)^2 = 100$
    $x + y = 14$

25. $xy + y + x = 17$
    $x^2y^2 + y^2 + x^2 = 129$

26. $b + a^2 = 5(a - b)$
    $a + b^2 = 2(a - b)$

27. $(13x)^2 + 2y^2 = 177$
    $(2y)^2 - 13x^2 = 3$

28. $x^2 + y^2 = 20$
    $5x^2 - 3y^2 = 28$

29. $x^2 = -5 - 3xy$
    $2xy = y^2 - 24$

30. $x + xy + y = 29$
    $x^2 + xy + y^2 = 61$

31. $x + y = 74$
$x^2 + y^2 = 3026$

32. $x^3 - y^3 = 37$
$x - y = 1$

33. $x^2 + y^2 - xy = 80$
$x - y - xy = -8$

34. $8a + 8b - ab - a^2 = 18$
$5a + 5b - b^2 - ab = 24$

35. $2y^2 - xy = 16$
$x^2 - xy - y^2 = 20$

36. $3u^2 + uv - v^2 = 9$
$v^2 - u^2 = 27$

37. $y = 1 - x$
$4x = y^2 + 2y - 7$

38. $x^2 - 4y^2 = 4$
$x^2 + 4y^2 = 16$

39. $4x^2 - xy = 1$
$16x^2 + y^2 = 10$

40. $x^2 - 3xy + y^2 = 2$
$2x^2 - xy + 2y^2 = 34$

41. $4x^2 + 4y^2 = 2a^2 + 2ab + b^2$
$8x^2 - 4y^2 = a^2 - 2ab - b^2$

42. $x + y = a$
$2x^2 + 2y^2 = a^2 + 1$

43. $(2x + y)(x - y - 1) = 0$
$4x^2 + 4xy + 8x - 4y = 3y^2$

44. $x^2 - 3xy + 2y^2 - 3(x - y) = 0$
$2x^2 - xy - y^2 - 9(x - y) = 0$

45. $(x + 2y)(x - y) - 3(x - y) = 0$
$(2x + y)(x + y) - 27(x + y) = 0$

46. $(2a - x)^2 - (2b - y)^2 = a^2 - b^2$
$(2a - x)(2b - y) = ab$

47. $2x^2 - 5xy + 3x - 2y = 22$
$5xy + 7x - 8y - 2x^2 = 8$

48. $7y^2 - 5x^2 + 20x + 13y - 27 = 0$
$5(x - 2)^2 - 7y^2 - 17y - 17 = 0$

49. $(3x + 4y)(7x - 2y) + 3x + 4y = 44$
$(3x + 4y)(7x - 2y) - 7x + 2y = 30$

50. $2(x + 4)^2 - 5(y - 7)^2 = 75$
$7(x + 4)^2 + 15(y - 7)^2 = 1075$

51. $\frac{1}{(x - 2)^2} + \frac{4}{(y + 2)^2} = 25$
$\frac{1}{x - 2} + \frac{2}{y + 2} = 7$

PROBLEMS

The two groups of problems, A and B, are of about equal difficulty. Either one may be used.

Group A

1. The hypotenuse of a right-angled triangle is 13 feet long and the area of the triangle is 30 square feet. Find the other sides.

2. If in problem 1 the hypotenuse is a feet long and the area is b square feet, find the sides of the triangle. Solve problem 1 by substituting in the formula obtained in this problem.

3. The sum of the squares of the two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the original number.

4. The sum of the reciprocals of two numbers is 3 and the product of the numbers is 9/14. Find the numbers.

5. The sum of the reciprocals of two numbers is s and the product of the numbers is t. Find the numbers. Solve problem 4 by substituting in the formula thus found.

6. A weight on one end of a lever balances a weight of 6 pounds placed 4 feet from the fulcrum on the other. If the unknown weight is moved 2 feet nearer the fulcrum, the weight balances 2 pounds placed 9 feet from the fulcrum on the other side. Find the unknown weight.

7. In the equation $x^2 + 2bx = 3$, the sum of the squares of the roots is 10. Find the value of the constant b. Substitute this value of b in the given equation and check that your value of b is correct.

8. A man is five times as old as his son, and the sum of the squares of their ages is equal to 2106. Find their ages.

9. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds three times the area of the smaller by 325 square feet. Find the length of their sides.

10. In problem 9 replace 100 and 325 by a and b, respectively, and solve the resulting problem. Then solve problem 9 by substituting in the formula that you obtain.

11. Two rectangles contain the same area, 480 square yards; the difference of their lengths is 10 yards, and of their breadths 4 yards. Find their sides.

12. In problem 11 replace 480, 10, and 4 by a, b, c, respectively, and solve the resulting problem. Then solve problem 11 by substituting in the formula that you obtain.

13. There is a number between 10 and 100; when multiplied by the digit on the left the product is 280; if the sum of the digits is multiplied by the same digit, the product is 55; find the number.

14. The area of a rectangle is a square feet and its perimeter is $2b$ feet. Find the length of its sides.

15. A picture measured inside the frame is a by b inches. The area of the frame is c square inches. Find its width.

16. Replace a, b, and c in problem 15 by ordinary numbers and solve the resulting problem by substituting in the formula that you have just obtained.

17. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2k$. Find by how much each side is increased.

18. In problem 17 replace a, b, $2k$ by ordinary numbers and solve the resulting problem by using the formula obtained in problem 17.

19. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the dimensions of the original piece of tin.

20. A rectangular piece of tin is a inches longer than it is wide. By cutting from each corner a square whose side is b inches and then turning up the sides, an open box containing c cubic inches is formed. Find the dimensions of the original piece of tin.

21. The difference of the cubes of two consecutive integers is 397. Find the integers.

22. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Group B

1. Find the unknown coefficient b in $2x^2 - bx + 9 = 0$, if it is given that the sum of the reciprocals of the roots is 1. Substitute the value of b that you find and check this value by solving the resulting equation.

2. Find two numbers the sum of whose squares is 74, and whose sum is 12.

3. Find two numbers the sum of whose squares is a, and whose sum is b. Then solve problem 2 by substituting in the formula that you obtain.

4. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes; the larger pipe would fill the cistern in 24 minutes less than the smaller one. Find the time taken by each.

5. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each for the opposite station. The train from A reaches B in nine hours, the train from B reaches A four hours after they meet; find the rate at which each train travels.

6. In problem 5 replace the given number by literal numbers and solve the resulting problem. Then solve problem 5 by substituting in the formula that you obtain.

7. The hypotenuse c and one side a are each increased by the same amount, thereby increasing the square on the other side by $2k$. How much was added to the hypotenuse?

8. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

9. Find two consecutive integers whose product is a.

10. In problems 7, 8, 9, replace the literal numbers by ordinary numbers and solve the resulting problems. Then solve your problems by substituting in the formulas obtained in solving problems 7, 8, 9.

11. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find a side of the square and the radius of the circle.

12. Find three consecutive integers such that the sum of their squares is a.

13. The area of a window is 2016 square inches and the inside length of the frame is 180 inches. Find the dimensions of the window.

14. Make a problem like the preceding using literal numbers. Solve this problem and then solve problem 14 by substituting in the formula that you obtain.

15. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk when measured on the side next the street is 560 yards. Find the dimensions of the block.

16. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow is 376 rods. Find the dimensions of the field.

17. A rectangular blackboard contains 38 square feet and the length of the molding is 27 feet. Find the dimensions of the board.

18. The difference between the sides of a right triangle is 8 and the hypotenuse is 42. Find the lengths of the sides.

19. Make a problem like the preceding using literal numbers. Solve and then find the answer for problem 18 by substituting in the formula.

20. An automobile running northward at the rate of 45 miles per hour is 60 miles south of the intersection with an east and west road. At the same time another automobile running westward on the crossroad at the rate of 50 miles per hour is 75 miles east of the crossing. How far apart (diagonally) will they be one hour later?

21. In problem 20, how long from the time of starting will these cars be 20 miles apart?

22. Find the area included in the main roads of a township if they are 4 rods wide. (In a township the main roads run along the section lines, one-half of the road on each side of the line.)

23. If the area included in the main roads of a township is 68,796 square rods, find the width of the roads.

CHAPTER 13:

RATIO AND PROPORTION; VARIATION

The words variation, ratio, and proportion are often used in describing mathematical relations, and it is necessary for us to learn the precise meaning of these words in order to understand what is said and written, and also in order to make ourselves understood. In the way of subject matter there is not much in this chapter that is absolutely new. The algebraic operations to be used are exactly those that we have studied heretofore.

164. Ratio.—Elaborate definitions of the word ratio have been given, but they all amount to this, that the ratio of a to b is the fraction a/b.[1] The Greeks used the concept "ratio" mainly in connection with their work on geometry, while their notion of a "fraction" was still very hazy, and developed an elaborate treatment of this ratio. Their idea of ratio has been passed on to us in connection with geometry and we have been a little slow in recognizing the plain fact stated above. The ratio a to b is often written $a : b$.

In speaking of the ratio a to b (a/b), the numerator of the fraction a/b is called the antecedent, and the denominator is called the consequent.

The law of fractions, $\frac{a}{b} = \frac{ma}{mb}$, is used when dealing with the fraction as a ratio. Thus if a city lot is 60 feet wide and 150 feet long, we say that the ratio of the width to the length is 60 to 150, or what is the same, this ratio is 2 to 5. When we say that the ratio of the dimensions of a rectangle is 2 to 5 we say nothing about the "size" of the rectangle, but we describe its "shape" completely.

$$\frac{60}{150} = \frac{2}{5}$$

[1] It is sometimes said that the ratio of two numbers indicates "the relation" between them, but this is too indefinite to be used as a definition. A relation between a and b may be that a is less than b or even that a is equal to b. The definition given in the text is perfectly definite and cannot well be made simpler than it is.

165. *Proportion.*—If two ratios are equal, they are said to form a proportion. In equation (1) at the right the quantities a, b, c, d form a proportion. This is also written $a : b :: c : d$ or $a : b = c : d$. In this proportion a and d are called the extremes, and b and c are called the means. The four quantities a, b, c, d are called the terms of the proportion.

$$\frac{a}{b} = \frac{c}{d} \quad (1)$$
$$ad = bc \quad (2)$$

The name "extremes" applied to a and d, and the name "means" applied to b and c no doubt come from the form $a : b = c : d$ rather than from the form $\frac{a}{b} = \frac{c}{d}$.

Clearing equation (1) of fractions gives equation (2), which states what is often called the fundamental property of a proportion. This property is usually given in words as follows.

In a proportion the product of the extremes is equal to the product of the means.

When three terms of a proportion are given, the remaining term can be found by solving equation (1) or (2) for that term.

In practice a ratio is usually reduced to a simple fraction in its lowest terms expressed in the same units. Thus, if a rug is 9 ft. 6 inches wide and 12 ft. 8 inches long, we say that the width and length are in the ratio 57 : 76, though the ratio $9\frac{1}{2} : 12\frac{2}{3}$ may at times be used.

166. *Compound ratios.*—Before making further use of proportion we shall develop the idea of a compound ratio. If the widths of two rectangles are the same, the ratio of their areas is equal to the ratio of their lengths. That is, if A_1 and A_2 are the areas and l_1 and l_2 the lengths, then $A_1/A_2 = l_1/l_2$. Hence, the areas and the lengths form a proportion. Again if the lengths are the same, the areas are in the same ratio as the widths. That is, $A_1/A_2 = w_1/w_2$. Then it follows that the areas A_1 and A_2 of any two rectangles are in the ratio represented by the product $l_1/l_2 \cdot w_1/w_2$, or l_1w_1/l_2w_2. The ratio $l_1w_1 : l_2w_2$ is said to be compounded of the ratios $l_1 : l_2$ and $w_1 : w_2$.

$$\frac{A_1}{A_2} = \frac{l_1}{l_2}$$
$$\frac{A_1}{A_2} = \frac{w_1}{w_2}$$
$$\frac{A_1}{A_2} = \frac{l_1w_1}{l_2w_2}$$

The idea of compound ratio is capable of many applications.

Example 1. The areas A_1 and A_2 of two triangles with the same base are in the same ratio as their altitudes h_1 and h_2; the areas of two triangles with the same altitude are in the same ratio as their bases, b_1 and b_2. Hence the ratio of the areas of any two triangles is given by the compound ratio $b_1h_1 : b_2h_2$.

$$\frac{A_1}{A_2} = \frac{h_1}{h_2} \qquad \frac{A_1}{A_2} = \frac{b_1}{b_2} \qquad \frac{A_1}{A_2} = \frac{b_1h_1}{b_2h_2}$$

Example 2. The volumes of two rectangular solids with the same base are in the same ratio as their altitudes h_1 and h_2; also the volumes of two such solids with equal altitudes are in the same ratio as the areas a_1b_1 and a_2b_2 of their bases. The ratio $a_1b_1h_1 : a_2b_2h_2$ may be regarded as compounded out of the three ratios $a_1 : a_2$, $b_1 : b_2$, and $h_1 : h_2$.

$$\frac{V_1}{V_2} = \frac{a_1b_1h_1}{a_2b_2h_2}$$

Example 3. The ratio of volumes of cylinders with the same base is equal to the ratio $h_1 : h_2$ of their altitudes. Also the ratio of the volumes of two cylinders with the same altitude is equal to the ratio $B_1 : B_2$ of the areas of their bases. If the bases are circles with radii r_1 and r_2, then the ratio of their areas is $r_1^2 : r_2^2$. Hence the ratio of the volumes of cylinders with circular bases whose radii are r_1 and r_2, and whose altitudes are h_1 and h_2 is $\dfrac{r_1^2h_1}{r_2^2h_2}$.

$$\frac{V_1}{V_2} = \frac{h_1}{h_2} \qquad \frac{V_1}{V_2} = \frac{B_1}{B_2} \qquad \frac{V_1}{V_2} = \frac{r_1^2h_1}{r_2^2h_2}$$

Note that the volumes V_1 and V_2 are not $r_1^2h_1$ and $r_2^2h_2$, but the *ratio* of these volumes is as given. Hence we find the ratio of these volumes though we may not know how to compute the area of a circle, which is really a much more difficult matter.

EXERCISES

1. Solve the proportion $a : b = c : d$ for each of the letters in terms of the other three.

2. Find the value of x in each of the following.

$$x : 4 = 7 : 3, \qquad 5 : x = 3 : 8, \qquad 4 : 9 = x : 5, \qquad 3 : 4 = 5 : x$$

3. The costs of city lots of the same width are in proportion to their lengths l_1 and l_2, and in proportion to their widths w_1 and w_2 if they have the same length. Express the ratio of the costs, C_1 and C_2, in terms of l_1, l_2, w_1, w_2.

4. The weights w_1 and w_2 of two cylindrical steel rods of the same radius are in proportion to their lengths l_1 and l_2. If the rods are of the same length the weights are in proportion to the squares of their radii, r_1^2 and r_2^2. Express the ratio of the weights of two such rods with lengths l_1 and l_2 and radii r_1 and r_2.

167. Continued proportion, transformation of a proportion.—A series of equal ratios is said to form a continued proportion.[1] Thus the expression at the right is a continued proportion. From such a proportion many relations among its terms can be found that are of value in many applications of mathematics, and especially in the study of geometry.

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$$

Theorem 1. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = \frac{a}{b}$.

PROOF: Let $\frac{a}{b} = \frac{c}{d} = k$. Then $a = bk$ and $c = dk$.

Adding, $a + c = (b + d)k$, or $k = \frac{a+c}{b+d}$. Since $\frac{a}{b} = k$, this proves the theorem.

Theorem 2. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$

then, any of these $= \left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{1/n}$

PROOF: $\left(\frac{a}{b}\right)^n = \frac{pa^n}{pb^n} = \frac{qc^n}{qd^n} = \frac{re^n}{rf^n} = \ldots$

Let $\frac{pa^n}{pb^n} = k$. Then $pa^n = kpb^n$, $qc^n = kqd^n$, . . .

or $$pa^n + qc^n + re^n \ldots = k(pb^n + qd^n + rf^n + \ldots)$$

and hence $$k = \frac{pa^n}{pb^n} = \frac{a^n}{b^n} = \frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots} = \frac{a^n}{b^n}$$

and therefore $$\frac{a}{b} = \left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{1/n}$$

If in this example n, p, q, r, . . . are all unity, then this formula reduces to the one shown below. This may be stated as follows.

In a series of equal fractions any one of the fractions is equal to the sum of the numerators divided by the sum of the denominators.

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots \quad \text{then } \frac{a}{b} = \frac{a + c + e + \ldots}{b + d + f + \ldots}$$

Or it may be stated:

In any continued proportion any antecedent is to its consequent as the sum of any number of antecedents is to the sum of their consequents.

[1] The expression "continued proportion" is sometimes restricted to the special case $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} =$, etc.

The theorem just stated is particularly useful in the study of geometry when we encounter such problems as that considered in §169. The Greeks, who developed a theory of ratio and proportion for use in geometry, dignified under the heads of theorems many simple relations among quantities that are in proportion, which we, with our more extensive understanding of algebra, might dispose of with simple remarks. Some examples are the following.

Given
$\frac{a}{b} = \frac{c}{d}$
Inversion
(1) $\frac{b}{a} = \frac{d}{c}$
Alternation
(2) $\frac{a}{c} = \frac{b}{d}$
Composition
(3) $\frac{a+b}{b} = \frac{c+d}{d}$
Division
(4) $\frac{a-b}{b} = \frac{c-d}{d}$
Composition and Division
(5) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

From the fundamental proportion $\frac{a}{b} = \frac{c}{d}$ we may prove the relations shown at the right. The names given to these are taken from texts on geometry.

To prove (1), divide 1 by both members, obtaining $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$. Then change $\frac{1}{\frac{a}{b}}$ to $\frac{b}{a}$ and $\frac{1}{\frac{c}{d}}$ to $\frac{d}{c}$.

To prove (2), write $ad = bc$ and divide both members by dc.

To prove (3), add 1 to both members, obtaining $\frac{a}{b} + 1 = \frac{c}{d} + 1 = \frac{a+b}{b} = \frac{c+d}{d}$.

To prove (4) subtract 1 instead of adding 1 as in (3).

To prove (5), divide the members of (3) by the members of (4).

EXERCISES

From $\frac{a}{b} = \frac{c}{d}$ show that

1. $\frac{pa - c}{pb - d} = \frac{a}{c}$
2. $\frac{pa + qb}{pa - qb} = \frac{pc + qd}{pc - qd}$.
3. $\frac{pa + qb}{pc + qd} = \frac{pa - qb}{pc - qd}$.
4. $\sqrt{\frac{pa^2 + qc^2}{pb^2 + qd^2}} = \sqrt[3]{\frac{pa^3 + qc^3}{pb^3 + qd^3}}$
5. $\sqrt{\frac{3a - 7c}{3b - 7d}} = \sqrt[4]{\frac{3a^4 - 7c^4}{3b^4 - 7d^4}}$

168. Fourth proportional; mean proportional.—If in a proportion $a : b = c : d$, b and c are equal, then this quantity is a mean proportional between a and d. If $a : c = c : d$, then d is a third proportional to a and c.

$$\frac{a}{c} = \frac{c}{d}$$
$$c^2 = ad$$
$$c = \sqrt{ad}$$
$$d = \frac{c^2}{a}$$

Some very old (and odd) expressions still linger in our mathematical literature: The ratio $a^2 : b^2$ is said to be the duplicate ratio of a and b, and $a^3 : b^3$ is the triplicate ratio of a and b. $\sqrt{a} : \sqrt{b}$ is the subduplicate ratio of a and b.

Sir Isaac Newton stated one of Kepler's laws of planetary motions in this language (as translated into English):

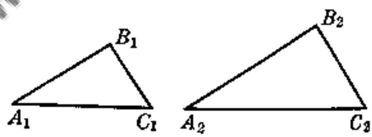
The triplicate ratio of the mean distances is equal to the duplicate ratio of the periods.

Exactly the substance of this statement is given in the proportion at the right.

$$\frac{d_1^3}{d_2^3} = \frac{p_1^2}{p_2^2}$$

169. A problem from geometry.—Our problem is based on two theorems:

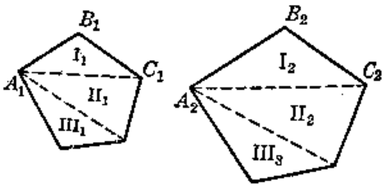
1. The ratio of the areas of two similar triangles is equal to the ratio of the square of two corresponding sides.



In the figure, $A_1B_1C_1$ and $A_2B_2C_2$ are similar triangles. Then

$$\frac{\text{Area}_1}{\text{Area}_2} = \frac{\overline{A_1B_1}^2}{\overline{A_2B_2}^2}$$

2. Two similar polygons can be divided into triangles that are similar in pairs.



The polygons at the right are similar and are divided into triangles I_1, I_2; II_1, II_2; III_1, III_2 so that these are similar in pairs. We want to show that the ratio of the areas of these polygons are in the ratio $\overline{A_1B_1}^2 : \overline{A_2B_2}^2$. Construct these figures, lettering them completely and prove this theorem. Remember that in similar polygons we have the continued proportion.

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1D_1}{C_2D_2}$$

This general theorem holds for similar polygons of any number of sides and hence can be extended to any similar figures (figures of the same shape or proportions) even if the figures are curved. That is,

The areas of figures of the same shape are in the same ratio as the squares of any two corresponding straight line segments.

This also holds for surfaces of similar solids. That is, *the area of the surfaces of similar solids is in the same ratio as the squares of corresponding line segments.*

A similar proposition holds for the volume of two similar solids, the ratio of the cubes of corresponding lines being used.

That is, *the ratio of the volumes of any two similar solids is equal to the ratio of the cubes of any two corresponding straight line segments.*

EXERCISES

Find the mean proportional for each of the following.

1. $2, 8$
2. $7, 346$
3. ax^2, a^3x^4
4. $(a-b)^4, a^2-2ab+b^2$
5. $\frac{1}{2}, \frac{1}{8}$
6. $\frac{1}{x}, \frac{1}{x^3}$
7. $\frac{1}{m^2n}, \frac{4}{m^2n^3}$
8. $a^2-b^2, (a+b)(a-b)^3$

Find a third proportional for each of the following.

9. $2, 3$
10. $7, 15$
11. $2x, 5x^2$
12. $(x-y), x^2-y^2$

Solve the following for x.

13. $3x-1 : 2x+4 = 3 : 7$
14. $5 : 9x-7 = 2x+9 : 12$
15. $8 : 37 = x-4 : 2x+9$
16. $7 : 3x+2 = 5 : 2x-3$
17. $\sqrt{x-1} : \sqrt{x+1} = 2 : 5$
18. $\sqrt{x+1} : \sqrt{x-1} = \sqrt{x+3} : \sqrt{x+2}$
19. $\frac{\sqrt{x^2+3}}{\sqrt{x-7}} = \frac{\sqrt{x^2+8}}{\sqrt{x+2}}$
20. $\frac{\sqrt{7}}{\sqrt{11}} = \frac{\sqrt{x^2-7x+2}}{\sqrt{x^2+x-6}}$
21. $\frac{x(x+1)}{x-4} = \frac{x^2+7}{x-3}$
22. $\frac{x^2(x+4)}{x+5} = x^2-6$

23. Find the ratio of the areas of two similar polygons of which two corresponding sides are 17 and 31.

24. Circles are similar plane figures and radii are corresponding dimensions (straight line segments). What is the ratio of the areas of two circles whose radii are 3 and 5½?

25. Spheres are similar solids and their diameters are corresponding lines. What is the ratio of the volumes of two spheres whose diameters are 2½ and 3?

170. Variation, direct variation.—Before making further use of proportion we shall introduce the subject of variation since this is often used in applications. As has been stated earlier (see page 69), a variable is a symbol that is free to represent any one of a certain set of numbers. Two variables may be connected by a relation that will determine the value of one of them when a value of the other is given. Such relations are usually expressed by means of equations.

When we say that the variable y varies directly as the variable x, we mean that these variables are subject to a relation such that any value of y divided by the corresponding value of x gives the same quotient for all pairs of values of x and y.

$$\frac{\text{Any value of } y}{\text{Corresponding value of } x} = \text{constant}$$

For the present this constant may be any fixed number whatever. The same relation may also be expressed by the equation $y = kx$, where k is a constant.

We have really used this idea in many simple examples, such as:

1. For a fixed rate of interest and a fixed principal the amount of interest earned varies directly as the time.
2. For a fixed rate and a fixed time, the interest varies directly as the principal.
3. For a fixed time and a fixed principal, the interest varies directly as the rate.
4. For a fixed speed, the distance varies directly as the time.
5. For a fixed time, the distance varies directly as the speed.

By letting p and r be fixed in the equation at the right, 1. above is expressed by this equation. By letting r and t be fixed or p and t be fixed, this equation will give 2. and 3. above.

$$i = prt$$

If s in the equation at the right is fixed, 4 is stated by this equation, and if t is fixed, 5 is stated by the equation.

$$d = st$$

If t is constant, then the equation $d = st$ states that d varies directly as s; while if s is a constant, then d varies directly as t. Similar remarks apply to $i = prt$, if any two of the quantities p, r, t are constants.

171. Joint variation.—The area of a rectangle varies jointly as the width and length, as does also the weight of a rectangular piece of metal of uniform thickness. If in this second case y is the weight and x and z the length and width, then $y = kxz$, where k is fixed, represents the variation.

$$y = kxz$$

For any metal plate of a given thickness k will be a constant, though the value of k will differ for different metals and different thicknesses.

If $y = kx^2$ (k still a constant), then y is said to vary directly as the square of x.

If $y = kx^3$, then y varies directly as x^3.

172. Inverse variations.—In many practical situations the product of two variables is a constant. Thus the volume v and the pressure p of a given quantity of gas with constant temperature vary so that the product $vp = k$, where k is constant. Then $p = k/v$ and $v = k/p$, and we say that p varies inversely as v and v varies inversely as p.

$$p = k/v$$
$$a = k/d^2$$

Again, the gravitational attraction between two bodies varies inversely as the square of the distance. If a is the attraction and d the distance, then $a = k/d^2$.

We also express the fact that one quantity Y varies as another quantity X by using the symbol $Y \propto X$.

Verbal Statement	Equation	Special Symbol
y varies directly as x	$y = kx$	$y \propto x$
y varies directly as x^2	$y = kx^2$	$y \propto x^2$
y varies directly as x^3	$y = kx^3$	$y \propto x^3$
y varies inversely as x	$y = k/x$	$y \propto 1/x$
y varies inversely as x^2	$y = k/x^2$	$y \propto 1/x^2$
y varies jointly as x and z	$y = kxz$	$y \propto xz$

We shall now give a series of examples involving variation.

Example 1. If y varies directly as x and if $y = 18$ for $x = 7$, find y for $x = 24$.

SOLUTION: $y = kx$ states that y varies directly as x. (2), (3), (4) state the remainder of the solution.

$$y = kx \quad (1)$$
$$18 = k \cdot 7 \quad (2)$$
$$k = \frac{18}{7} \quad (3)$$
$$y = \frac{18}{7} \cdot 24 \quad (4)$$

Example 2. If z varies jointly as x and y and if $z = 48$ when $x = 3$ and $y = 4$, find z when $x = 6$ and $y = 5$.

$$\begin{aligned} z &= kxy && (1)\\ 48 &= k \cdot 3 \cdot 4 && (2)\\ z &= k \cdot 6 \cdot 5 && (3)\\ \frac{z}{48} &= \frac{k \cdot 6 \cdot 5}{k \cdot 3 \cdot 4} && (4)\\ &= \frac{5}{2} && (4)\\ z &= 120 \end{aligned}$$

Explain the steps in the solution at the right. Note that k is canceled out and its value is not found.

The problem may also be solved by finding the value of k from (2) and substituting in (3). In many problems this is the easier method.

Example 3. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 18 inches if a piece with radius 7 inches weighs 11 pounds.

$$\begin{aligned} w &= kr^2 && (1)\\ 11 &= k \cdot 7^2 && (2)\\ w &= k \cdot 18^2 && (3) \end{aligned}$$

Example 4. The distance that a body falls toward the earth in one second under the attraction of gravity varies inversely as the square of its distance from the earth's center. A body falls 16.08 feet in one second at the earth's surface, which is 4000 miles from its center. How far will it fall in one second at a distance of 240,000, the distance of the moon?

$$\begin{aligned} s &= \frac{k}{d^2}\\ 16.08 &= \frac{k}{4000^2}\\ s &= \frac{k}{240{,}000^2}\\ s &= 16.08 \cdot \frac{1}{60^2}\\ &= .0044 \end{aligned}$$

Explain the equations at the right, eliminate k, and check the computation.

Why is the value of s so much smaller at the distance of the moon than it is at the earth's surface?

This is one of the problems that Sir Isaac Newton had to solve in his effort to prove that the attraction of gravity accounts for the motion of the moon if that attraction varies inversely as the square of the distance.

Example 5. The law of gravitation, often called Newton's law is: The attraction between any two masses of matter m_1 and m_2 varies jointly as the product of the masses and inversely as the square of the distance. Let d represent the distance and a the gravitational attraction, and state this law in the form of an equation.

$$a = \frac{km_1m_2}{d^2}$$

Is the equation at the right correct? This short equation has sometimes been said to be the most amazing statement ever written.

PROBLEMS

1. The weight of a triangle cut from a steel plate of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and the base 6.

2. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 feet, if a piece of radius 7 feet weighs 196 pounds.

3. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of 10 seconds.

4. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

5. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. How much will this increase the amount of heat received?

6. The weights of bodies of the same shape and the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 ounces, how much will a ball of the same material weigh whose radius is $3\frac{1}{2}$ inches?

7. The weight of a body above the earth's surface varies inversely as the square of its distance from the earth's center. If an object weighs 2000 pounds at the earth's surface, what would be its weight if it were 12,000 miles above the center of the earth, the radius of the earth being 4000 miles?

8. The distance fallen by a body, starting from a position of rest in a vacuum near the earth's surface, is proportional to the square of the number of seconds occupied in falling. If a body falls 256 feet in 4 seconds, how far will it fall in 8 seconds?

9. The kinetic energy E of a mass of m pounds moving with a velocity v, is proportional to mv^2. If $E = 2500$ foot-pounds when a body weighing 64 pounds is moving at a velocity of 50 feet per second, find the kinetic energy of a body weighing 20 pounds whose velocity is 300 feet per second.

10. If one body is sliding on another, the force of sliding friction is proportional to the normal (perpendicular) pressure between the bodies (if this pressure is moderate). If the sliding friction between two cast iron plates is 60 pounds when the normal pressure is 270 pounds, find the sliding friction when the normal pressure is 575 pounds.

11. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. If a wire 350 feet long and 3 millimeters in diameter has a resistance of 1.08 ohms, find the length of a wire of the same material whose resistance is .81 ohm and diameter is 2 millimeters.

12. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cubic feet when under pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

13. The approximate amount of steam per second which will flow through a hole varies jointly as the steam pressure and the area of a cross section of the hole. If 40 pounds of steam per second at a pressure of 200 pounds per square inch flow through a hole whose area is 14 square inches, how much steam at a pressure of 250 pounds per square inch will flow through a hole whose area is 20 square inches?

14. The maximum safe load of a horizontal beam supported at its ends varies directly as its breadth and the square of its depth and inversely as the distance between the supports. If the maximum is 2400 pounds for a beam 4 inches wide and 10 inches deep, with supports 15 feet apart, find the maximum load for a beam of the same material which is 3 inches wide and 5 inches deep, with supports 20 feet apart.

15. The illumination received from a source of light varies inversely as the square of the distance from the source, and directly as its candle power. At what distance from a 50 candle-power light would the illumination be one-half that received at 20 feet from a 40 candle-power light?

16. The current in an electric circuit varies directly as the electromotive force and inversely as the resistance. In a certain circuit, the electromotive force is A volts, the resistance is b ohms, and the current is c amperes. If the resistance is increased by 20%, what percentage of increase must occur in the voltage to increase the current by 30%?

17. Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square feet when the radius is 7 feet; find the area of a circle whose radius is 10 feet 6 inches.

18. The area of a circle varies as the square of its diameter; prove that the area of a circle whose diameter is $2\frac{1}{2}$ inches is equal to the sum of the areas of two circles whose diameters are $1\frac{1}{2}$ and 2 inches respectively.

19. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving at the rate of 15 miles per hour; find the velocity of the wind when the pressure on a square yard is 16 pounds.

20. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same; it also varies directly as its thickness while its diameter remains the same. The diameters of two silver coins are in the ratio of 4:3. Find the ratio of their thickness if the value of the first is four times that of the second.

21. Newton's Law of Gravitation states that the force with which each of two masses of m pounds and M pounds attracts the other varies directly as the product of the masses and inversely as the square of the distance between the masses. Find the ratio of the force of attraction when two masses are 6000 miles apart to the force when they are 2000 miles apart.

CHAPTER 14:

LOGARITHMS AND THEIR USES

In many applications of mathematics, such as in surveying, engineering, navigation, astronomy, and in problems related to finance (business), long computations occur frequently. Such work is shortened greatly by the use of logarithms, which enable us to replace multiplication and division by addition and subtraction. In this chapter we shall make a brief study of logarithms and their uses.

173. *Definition of a logarithm.*—The index of the power to which a given number, called the base, must be raised to equal a second number, is called the logarithm of the second number.

That is, x is the logarithm of N to the base a if

$$a^x = N \text{ (exponential notation).}$$

This is also denoted by writing,

$$x = \log_a N \text{ (logarithmic notation).}$$

Thus, since $2^4 = 16$, $4 = \log_2 16$; since $10^2 = 100$, $2 = \log_{10} 100$; since $10^3 = 1000$, $3 = \log_{10} 1000$.

In developing a table of logarithms the same base must be used throughout. While any positive number except 1 might be used as a base, in tables for practical computing 10 is invariably used.

174. *Systems of logarithms.*—The system of logarithms having 10 as its base is called the Common System of Logarithms. Another system, called the Napierian or Natural System, is used for theoretical purposes.

In this chapter the common system only is used. Since the base of all logarithms in that system is the same, it is omitted in the notation. Thus "log N" is understood to mean "$\log_{10} N$".

175. *Laws of exponents.*—Since logarithms are exponents they have the properties (that is, they obey the laws) of exponents (see pages 34 and 114). The following laws of exponents are needed in developing the rules for logarithmic computation.

(1) $a^x \cdot a^y = a^{x+y}$ (4) $\sqrt[x]{a} = a^{1/x}$

(2) $a^x \div a^y = a^{x-y}$ (5) $a^{-x} = \dfrac{1}{a^x}$

(3) $(a^x)^y = a^{xy}$ (6) $a^0 = 1$

176. *Logarithm of a product.*—The logarithm of a product is the sum of the logarithms of the factors.

Take 10 as the base and assume that

$$M = 10^x,\ N = 10^y.$$

Then, $\log M = x$, $\log N = y$.
By the laws of exponents, $MN = 10^x \cdot 10^y = 10^{x+y}$.

Hence, $x + y = \log MN$.

That is, $$\log (MN) = \log M + \log N.$$

Similarly, if in addition to the above $P = 10^z$,

then $$MNP = 10^x \cdot 10^y \cdot 10^z = 10^{x+y+z}.$$

Hence, $$\log (MNP) = \log M + \log N + \log P.$$

We therefore have the rule:

To find the logarithm of a product, find the sum of the logarithms of the factors.

177. *Logarithm of a quotient.*—The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.

If M and N are the same as in the preceding section, we have

$$\frac{M}{N} = \frac{10^x}{10^y} = 10^{x-y}, \text{ or } \log \frac{M}{N} = x - y.$$

That is, $$\log \frac{M}{N} = \log M - \log N.$$

Hence we have the rule:

To find the logarithm of a quotient, subtract the logarithm of the divisor from the logarithm of the dividend.

178. Logarithm of a power.—To find the logarithm of a power of a number proceed as follows.

If $M = 10^x$, then $M^y = (10^x)^y = 10^{xy}$.
Hence, the logarithm of M^y is xy

or,
$$\log M^y = y \log M,$$

and we have the rule:

To find the logarithm of a power of a number, multiply the logarithm of the number by the index of the power.

179. Logarithm of a root.—The rule of the preceding paragraph holds also when the index of the power is a fraction.

By (4) of §175, $\sqrt[q]{M} = M^{1/q}$. Hence by §178,
$$\log \sqrt[q]{M} = \log M^{1/q} = \frac{1}{q} \log M,$$

and we have the rule:

To find the logarithm of a root of a number, divide the logarithm of the number by the index of the root.

It follows that in general $\log M^{p/q} = p/q \log M$.

180. Zero and negative logarithms.—From (6) §175, $10^0 = 1$.

Hence,
$$\log 1 = 0,$$

and
$$\log \frac{1}{M} = \log 1 - \log M = -\log M.$$

Since 10^x is greater than 1 for every positive value of x (integral or fractional) it follows that:

The logarithm of a positive number less than 1 is negative.

SIGHT WORK

1. Given $\log 43 = 1.6335$ and $\log 76 = 1.8808$, find $\log 43 \times 76$, also $\log 76 \div 43$, and $\log \sqrt{43 \cdot 76}$.

2. Given $\log 21 = 1.3222$, $\log 186 = 2.2695$, and $\log 324 = 2.5105$, find $\log 21 \times 186 \times 324$.

3. From the logarithms given in Example 2, find $\log \frac{324 \times 186}{21}$.

SUGGESTION: First find $\log 324 \times 186$.

4. Given, $\log 2147 = 3.3318$ find $\log \sqrt[4]{2147^3}$.

181. Approximate logarithms.—The logarithms of all rational numbers other than 1, 10, 100, 1000, . . . , and their reciprocals, can be given only approximately by means of decimals or common fractions. The closeness of the approximation depends upon the number of decimal places to which a table is carried.

Different tables are used for different purposes. Thus there are tables with four, five, six, seven, ten, fourteen, and even twenty places.

182. Characteristic and mantissa.—The integral part of a logarithm is called the characteristic and the decimal part is called the mantissa.

Thus, in the logarithm 3.7812, 3 is the characteristic and .7812 is the mantissa.

183. Rules for the characteristic.—From the following table some important inferences may be drawn. Note that the equations in the second column are immediate consequences of those in the first.

$10^4 = 10{,}000$	$\log 10{,}000 = 4$
$10^3 = 1{,}000$	$\log 1{,}000 = 3$
$10^2 = 100$	$\log 100 = 2$
$10^1 = 10$	$\log 10 = 1$
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = .1$	$\log .1 = -1$
$10^{-2} = .01$	$\log .01 = -2$
$10^{-3} = .001$	$\log .001 = -3$
$10^{-4} = .0001$	$\log .0001 = -4$

From this table it is evident that if a number lies between 1,000 and 10,000, its logarithm lies between 3 and 4, and may be written as 3 + a fraction. Such a number has four digits preceding the decimal point. Similarly, a number of 3 digits lies between 100 and 1,000 and its logarithm is 2 + a fraction. Again, the logarithm of a number of 1 digit is 0 + a fraction.

Hence we have:

Rule I. If a number is greater than 1, the characteristic of its logarithm is positive[1] *and is one less than the number of digits preceding the decimal point.*

[1] Zero is here regarded as a positive number.

Again, from the table it is evident that if a number lies between 1 and .1, its logarithm lies between -1 and 0, and may be written as $-1 +$ a fraction. For a number lying between .1 and .01, the logarithm is between -2 and -1, and may be written as $-2 +$ a fraction. Now, any number between .1 and .01, as .0831 or .01001, has one zero immediately following the decimal point, while a number between 1 and .1 has no zero immediately following the decimal point.

Continuing in this way, it appears that the characteristic of the logarithm of such a number having two zeros immediately following the decimal point is -3; for one having three such zeros, it is -4, and so on.

Hence we have:

Rule II. The characteristic of the logarithm of a decimal fraction is negative, and is numerically one greater than the number of zeros immediately following the decimal point.

From Rules I and II the characteristic of any number written in the decimal form may be found at once.

184. Ways of writing negative characteristics.—From the table of logarithms, which is a table of mantissas, it is found that the mantissa of the logarithm of 347 is .5403. From Rule I above, the characteristic is 2. Hence,

$$\log 347 = 2.5403.$$

If, however, we desire to write the logarithm of .0347, we must recognize that the logarithm is the characteristic -2 (by Rule II above) plus a positive fraction (in this case .5403 again). It would plainly be misleading to write this, -2.5403. It may be written $.5403 - 2$, but is more often written $\bar{2}.5403$, the minus sign being written above the 2 to indicate that only the 2, and not the decimal, is affected.

In computation a negative characteristic is often replaced by some integer minus 10.

Thus, since $-2 = 8 - 10$,

$$\begin{aligned} \log .0347 &= .5403 - 2 \\ &= .5403 + 8 - 10 = 8.5403 - 10. \end{aligned}$$

Likewise -1 is replaced by $9 - 10$, as in

$$.2780 - 1 = \bar{1}.2780 = 9.2780 - 10.$$

185. The significant part of a number.—If all zeros at the beginning and the end of a number, and also its decimal point, are omitted, the remaining part is called the significant part of the number.

Thus the significant part of 63800 is 638, the significant part of .0012 is 12, the significant part of 0.014 is 14, the significant part of 48 is 48, and the significant part of 3.41 is 341.

186. Numbers having the same mantissa.—From §184 we have

$$\log 347 = 2.5403.$$

Since $3470 = 347 \times 10,$

it follows that, $\log 3470 = \log 347 + \log 10$
$= 2.5403 + 1 = 3.5403.$

Also $34.7 = 347 \div 10$

and hence, $\log 34.7 = \log 347 - \log 10$
$= 2.5403 - 1 = 1.5403.$

In this manner the following table is easily constructed.

$\log 3.47 = 0.5403$	$\log .347 = \bar{1}.5403$
$\log 34.7 = 1.5403$	$\log .0347 = \bar{2}.5403$
$\log 347 = 2.5403$	$\log .00347 = \bar{3}.5403$
$\log 3470 = 3.5403$	$\log .000347 = \bar{4}.5403$
$\log 34700 = 4.5403$	$\log .0000347 = \bar{5}.5403$

Hence we conclude:

The logarithms of numbers having the same significant parts have the same mantissa.

SIGHT WORK

1. Given, $\log 1784 = 3.2514$, find the logarithms of 178.4, 17.84, 1.784, 0.1784, 17840.
2. Given, $\log 74.96 = 1.8749$, find the logarithms of .7496, 7496, 7.496, 749.6, .07496, 74960.

187. Finding logarithms of numbers; interpolation.—In the table on pages 418, 419 the mantissas of numbers with three significant figures are found directly. The first two figures of the numbers are given in the first column to the left, and the third figure is in the first line of the page.

Thus, to find log 376 we go down the first column at the left to the number 37 and then across the page to the column marked 6 at the top, where we find 5752, which is the mantissa of the required logarithm. Since we know by §183 that the characteristic is 2, we have log 376 = 2.5752.

In this manner the logarithm of a number with three (or less) significant figures is read directly from the table.

If we wish to find log 3763, we note that the mantissas corresponding to 376 and 377 are 5752 and 5763 respectively.

Hence,
$$\log 3770 = 3.5763$$
$$\log 3760 = 3.5752.$$

The difference between these mantissas is 11. Since 3763 is three-tenths of the way from 3760 to 3770 we add $.3 \times 11 = 3.3$ to the mantissa of 3760, obtaining 5755 as the mantissa of 3763. Since the fractional part of this mantissa is less than .5 it is omitted.

To find the mantissa for 3768 we should add $.8 \times 11 = 8.8$ or 9, giving 5761.

Hence log 3763 = 3.5755 and log 3768 = 3.5761.

The process used in finding the logarithms of numbers with more than three significant figures by means of this table is called interpolation.

In working with a four-place table the figures in the mantissas are not to be extended beyond that number.

SIGHT WORK

1. Find log 780, log 43.1, log 9.24, log .0781.
2. Find log 49.16, log 8292, log 3.164, log .1987.
3. Find log 8.197, log .0089, log 7, log 81.07.
4. Find log 365.4, log .0341, log 8.75, log 3416.
5. Find log 5970, log 832, log .0015, log 41.43.
6. Find log 3.265, log 5910, log 3.257, log .8239.
7. Find log 44.21, log 369.5, log 398.1, log .5436.
8. Find log 9482, log .0157, log 87.59, log 41.5.
9. Find log 817.4, log 32.42, log 1.462, log .0247.
10. Find log 31.97, log 4.731, log 0.9143, log 2.056.
11. Find log 307.4, log 9.009, log 5.017, log 1.097.
12. Find log 3107, log 310.7, log 31.07, log 3.107.
13. Find log 5.341, log .5341, log .05341, log .005341.

188. ***Antilogarithms.***—A number is called the antilogarithm (usually written and pronounced "antilog") of its logarithm.

Thus, if $a = \log N$, then $N = \text{antilog } a$.

If a given mantissa occurs exactly in the table, the corresponding number (antilog) is found at once.

Thus, if the given mantissa is 9042, we know that the significant part of the antilog is 802.

When the given mantissa does not occur exactly in the table, the antilog must be found by interpolation as in the following example.

Example. Find antilog 1.8104.

SOLUTION: We first find the significant part of the number. For this purpose the characteristic is disregarded. In the table we find that the mantissas next below and next above the given mantissa 8104, are 8102 and 8109, and that the numbers corresponding to these are 646 and 647. Moreover, 8104 is two-sevenths of the way from 8102 to 8109.

Since $2 \div 7 = .29$ we annex 29 at the right of 646, obtaining 64629 as the significant part of antilog 1.8104. Since we are using a four-place table we write this significant part 6463. The characteristic being 1, we have

$$\text{antilog } 1.8104 = 64.63.$$

EXERCISES

Find the antilogarithms of the following.

1. 1.6542
2. 2.6667
3. 3.6777
4. 0.6831
5. $\bar{1}.6917$
6. 1.6539
7. 2.6719
8. $\bar{1}.6803$
9. $\bar{2}.6913$
10. 0.6649
11. 1.6690
12. 0.6720
13. 0.6910
14. $\bar{2}.6800$
15. $\bar{3}.6549$
16. 0.6901
17. $\bar{4}.6720$
18. $\bar{2}.6830$
19. $\bar{2}.6709$
20. $\bar{1}.6671$
21. $\bar{2}.8732$
22. 1.4107
23. $\bar{2}.6732$
24. 0.2527
25. $\bar{1}.3980$
26. $\bar{3}.2097$
27. 2.9340
28. $\bar{1}.5432$

In using logarithms it is necessary to add, subtract, multiply, and divide them. In doing this it is necessary to take account of the following facts.

(*a*) *The mantissa is always positive.*
(*b*) *The characteristic is always an integer.*

In the study of trigonometry where logarithms are used extensively, this is treated more fully.

189. ***Computation by means of logarithms.***—The following examples illustrate the use of logarithms and the form into which the work should be put.

Since there are no logarithms of negative numbers, products (and quotients) of negative numbers are obtained by treating them as if they were positive and then prefixing the proper sign in the result.

Example 1. Find the product of 49.6×82.7.

SOLUTION:

$$\begin{aligned} \log 49.6 &= 1.6955 \\ \log 82.7 &= \underline{1.9175} \\ & \ 3.6130 = \log 4102. \end{aligned}$$

On multiplying we should find that the product is 4101.92. Hence the result is correct to four significant figures, which is about the limit of accuracy when a four-place table is used.

Example 2. Find the value of $\dfrac{14.64 \times 811.2}{61.34}$.

SOLUTION:

$$\begin{aligned} \log 14.64 &= 1.1656 \\ \log 811.2 &= \underline{2.9091} \\ & \ 4.0747 \\ \log 61.34 &= \underline{1.7878} \\ & \ 2.2869 = \log 193.6. \end{aligned}$$

Example 3. Find the value of $9.47^2 \times 1.19^3$.

SOLUTION:

$$\begin{aligned} \log 9.47^2 &= 1.9526 \\ \log 1.19^3 &= \underline{0.2265} \\ & \ 2.1791 = \log 151.0. \end{aligned}$$

Writing down $\log 9.47^2 = 1.9526$ involves multiplying the logarithm of 9.47 by 2. This should be done mentally and no numbers should be written down, except those shown here.

Example 4. Find the value of $21.3^2 \times 64.7^3 \div 34.1^4$.

SOLUTION:

$$\begin{aligned} \log 21.3^2 &= 2.6568 \\ \log 64.7^3 &= \underline{5.4327} \\ & \ 8.0895 \\ \log 34.1^4 &= \underline{6.1312} \\ & \ 1.9583 = \log 90.84. \end{aligned}$$

Example 5. Find the value of $\sqrt{\dfrac{17.4 \times 81.6}{24.3^2}}$

SOLUTION:

$$\begin{aligned} \log 17.4 &= 1.2405 \\ \log 81.6 &= \underline{1.9117} \\ & 3.1522 \\ \log 24.3^2 &= \underline{2.7712} \\ 2)\, & .3810 \\ & .1905 = \log 1.551. \end{aligned}$$

EXERCISES

Find the values of the following correct to four significant figures.

Group A

1. $\dfrac{31.9 \times 7.3}{6.4 \times 319}$
2. $\dfrac{5.7 \times \sqrt{4.7}}{9.8 \times 5.3}$
3. $\dfrac{6.8 \times \sqrt{8.9}}{\sqrt{4.7 \times 37}}$
4. $\dfrac{65.4 \times 37.8}{4.56 \times 78.3}$
5. $\dfrac{\sqrt[3]{6.54} \times 38.7}{54.6 \times 84.7}$
6. $\dfrac{\sqrt{6.58} \times \sqrt{41.7}}{\sqrt{81.7} \times \sqrt{8.7}}$
7. $\dfrac{92.7 \times 34.1}{16.4 \times 27 \times 8}$
8. $\dfrac{64.1 \times 72 \times 6}{13 \times 47 \times 61}$
9. $\dfrac{31 \times 74 \times 16.9}{14.6 \times 27 \times 12}$
10. $\dfrac{19.8 \times 72.4 \times 31.7}{98.1 \times 24.7 \times 17.3}$
11. $\dfrac{640 \times .0139}{.024 \times 71.6 \times .028}$
12. $\dfrac{46 \times 319 \times 47}{71.6 \times 34.7 \times 21}$
13. $\dfrac{37.8 \times 78.3}{83.7 \times 7.38}$
14. $\dfrac{34.6 \times 27.8}{45.3 \times 78.2}$
15. $\dfrac{49.8 \times 26.4 \times 17}{34.2 \times 71.5 \times 19}$

Group B

1. $\dfrac{38.42 \times 6.54}{74.78 \times 913}$
2. $\dfrac{7.76 \times 67.8}{\sqrt{89.3} \times \sqrt{874}}$
3. $\dfrac{38.62 \times 29.42}{73 \times .191 \times 871}$
4. $\dfrac{49.3 \times 34.8}{84.8 \times 2.64}$
5. $\dfrac{13.4 \times 6.43}{3.41 \times 26.4}$
6. $\dfrac{62.8 \times 26.8}{39 \times 76 \times 84}$
7. $\dfrac{7.31 \times 49.6}{81 \times 42 \times 73}$
8. $\dfrac{67.72 \times 84.9}{7.14 \times 27 \times 43}$
9. $\dfrac{\sqrt{71.4 \times 26.8}}{\sqrt[3]{93.8 \times 176}}$
10. $\dfrac{3876}{\sqrt{2} \times \sqrt{3} \times \sqrt{8760}}$
11. $\dfrac{61.9 \times 847}{39 \times 8.7 \times 64.8}$
12. $\dfrac{74.8 \times 48.7}{32.1 \times 13.2}$
13. $\dfrac{37.62 \times 47.8}{27.3 \times 14 \times 89}$
14. $\dfrac{.450 \times .0174}{.045 \times 87 \times .0072}$
15. $\dfrac{19.84 \times 74.7}{34.7 \times 26 \times 92}$

190. Exponential equations.—An equation of the type $a = b^x$ cannot be solved for x by any method we have studied thus far. By using logarithms this equation may be solved as shown at the right. The quotient $\log a \div \log b$ may be found by ordinary division.

$$a = b^x$$
$$\log a = x \log b$$
$$x = \frac{\log a}{\log b}$$

Example. Solve for x. $7 = 2^x$.

SOLUTION: $x = \log 7 \div \log 2 = 0.8451 \div 0.3010 = 2.808$.

EXERCISES

Solve for x.

1. $15 = 3^x$
2. $24 = (2.06)^x$
3. $117 = (1.03)^x$
4. $3 = (1.05)^x$
5. $7(1.052)^x = 15$
6. $9(1.047)^x = 14.5$

191. Solving problems by means of logarithms.—Some problems that can be solved without logarithms can be solved more easily by using them. Other problems that are simple when logarithms are used are very difficult, or almost impossible, without them. Both of these types are illustrated in the problems below.

Example. Solve $15 = x^{11}$

SOLUTION: $\log 15 = 11 \cdot \log x$, $\log x = \frac{\log 15}{11}$, $x = 1.2791$

This Example would be *very* laborious without the use of logarithms, and the one in §190 almost impossible.

Problem 1. If \$1 is invested at 3% interest compounded annually, what is the amount in 10 years?

We need to find the value of $(1.03)^{10}$. By comparison with page 208, the result at the right is seen to be correct to five significant figures.

$$\log 1.03 = 0.01284$$
$$10 \log 1.03 = 0.12840$$
$$\text{antilog } .12840 = 1.3440$$

Problem 2. At 3% interest compounded annually, in how long will an investment double itself? Let t be the required number of years. If \$1 is to double itself at 3% in t years, then $(1.03)^t = 2$. This equation is solved at the right.

$$2 = (1.03)^t$$
$$\log 2 = t \log 1.03$$
$$t = \frac{\log 2}{\log 1.03} = \frac{.30103}{0.01284} = 23.44$$

The same result may be obtained by interpolating in the table on page 213. A direct solution not using logarithms would be very troublesome.

Problem 3. A man bought a famous picture for \$35,000 and 15 years later sold it for \$75,000. What rate per cent interest compounded annually did he make on the investment?

Under the conditions of the problem $35{,}000(1 + i)^{15} = 75{,}000$, i being the rate per cent expressed as a decimal. Check the solution at the right. Explain each step.

$$
\begin{aligned}
35{,}000(1 + i)^{15} &= 75{,}000 \\
(1 + i)^{15} &= \frac{15}{7} \\
15 \log (1 + i) &= \log 15 - \log 7 \\
\log (1 + i) &= \frac{\log 15 - \log 7}{15} \\
&= .02203 \\
1 + i &= 1.052 \\
\therefore i &= .052 = 5.2\%
\end{aligned}
$$

PROBLEMS

Solve by using logarithms.

1. How many cubic feet of air are there in a room whose dimensions in feet are 25.8 by 62.3 by 14.6?

2. How many liquid gallons can be contained in a right circular cylinder whose height is 140 feet and whose base has a radius of 46.4 feet? (One gallon equals 231 cubic inches.)

3. Find the amount of \$100 in 15 years, allowing compound interest at 4 per cent per annum.

4. Find in how many years a sum of money will amount to one hundred times its value at 5½ per cent per annum compound interest.

5. If in the year 1600 a sum of \$1000 had been left to accumulate for 300 years, find its amount in the year 1900, reckoning compound interest at 4 per cent per annum.

6. $\dfrac{\sqrt[3]{442.6} \times \sqrt[4]{.002} \times \sqrt[3]{4.8}}{(18)^2 \times .7^3 \times (3.4562)^{1/2}}$

7. $\dfrac{24.051 \times .02456}{.006705 \times .0203}$

8. $\dfrac{56.73\sqrt{152}}{(21.832)^3}$

9. $\left(\dfrac{25.6\sqrt{2.85}}{.263184}\right)^{1/2}$

10. $\dfrac{17.9(1.03)^4}{4647(10^{2.39})}$

11. $\dfrac{15^{12}\sqrt{.89}}{(.571372)^2}$

12. $\left(\dfrac{15.7(.68)^2}{45.161}\right)^{2/3}$

13. $\left(\dfrac{6731.84}{15.2\sqrt{.896}}\right)^{3/4}$

14. $\dfrac{\sqrt{853.46}}{10^{3.64}(.0138)}$

15. $\dfrac{\sqrt{57.84} - 2.63}{85 - \sqrt{.1986}}$

16. $\dfrac{\sqrt[9]{105}}{\sqrt[13]{76}}$

17. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$

18. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt[5]{.83} \times \sqrt{23}}}$

19. $\dfrac{\sqrt{9.8149} \times 80.8008}{\sqrt[7]{8283} \times (.0006412)^4}$

20. $\sqrt[3]{\dfrac{12.876 \times \sqrt{.068} \times (.015)^2}{29.029 \times (52.81)^4 \times (.4)^9}}$

CHAPTER 15:

ELEMENTARY SERIES; INVESTMENTS

If we arrange a number of objects in order so that we have a first, a second, a third, and so on, we say we have a sequence. Our most familiar sequence is the numbers that we use when we count. In this chapter we shall study two important types of sequences of numbers, and the sums we obtain when the terms of these sequences are added. We shall then use such sums in solving some of the fundamental problems on investments. Note that in a sequence, as just defined, the successive numbers need not be related in any way. However, in practical work they are connected by some definite relation, as is the case in the sequences considered below.

192. Arithmetic sequence; arithmetic series.—The numbers used in forming a sequence are called the terms of the sequence. If we start with any number a, then add a number d (positive or negative) to form a second term $a + d$, then add d again to form a third term $a + 2d$, and so on, we have an arithmetic sequence. If there are n terms in this sequence, then the last term is $a + (n - 1)d$. The sum of the terms of this sequence is an arithmetic series. The "arithmetic series" is also called an "arithmetic progression" and may be denoted A.P. Note the difference between a "sequence" and a "series." The series is the sum of the terms of the sequence. The standard form of an arithmetic series is:

$$a + [a + d] + [a + 2d] + [a + 3d] + \dots + [a + (n - 1)d]$$

The sum of the first n positive integers is an arithmetic series in which $a = 1$, $d = 1$, $n = n$.

EXERCISES

1. Write an arithmetic sequence in which $a = 2$, $d = 3$, $n = 6$. Write the arithmetic series consisting of the same terms.
2. Write an arithmetic series in which $a = 4$, $d = \frac{1}{4}$, and $n = 8$.
3. Write an arithmetic series in which $a = 10$, $d = -1$, $n = 10$.
4. Write an arithmetic series in which $a = 5$, $d = -2$, $n = 7$.

193. Sum of an A.P.—The arithmetic progression is given in terms of a, d, and n. Our present problem is to find a simple formula in terms of these letters by means of which the sum indicated in the progression can be computed. Denote the last term of an arithmetic progression by l and its sum by S. Then

$$S = a + [a + d] + [a + 2d] + \dots + [a + (n - 2)d] + [a + (n - 1)d]$$

Note that $l = a + (n - 1)d$, $l - d = a + (n - 2)d$, $l - 2d = a - (n - 3)d$. Using these values in writing the last terms of the series, and then writing the terms in the reverse order, we have equations (1) and (2).

$$S = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l \qquad (1)$$
$$S = l + (l - d) + (l - 2d) + \dots + (a + 2d) + (a + d) + a \qquad (2)$$
$$2S = (a + l) + (a + l) + (a + l) + \dots + (a + l) + (a + l) + (a + l)$$

Adding the terms as they stand in columns, we find that each sum is $a + l$. Since there are n terms in the series and hence n sums $(a + l)$, it follows that $2S = n(a + l)$, and therefore $S = \frac{n}{2}(a + l)$. If we use $l = a + (n - 1)d$, then $S = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$.

$$l = a + (n - 1)d \qquad \text{I}$$
$$S = \frac{n}{2}(a + l) \qquad \text{II}$$
$$S = \frac{n}{2}[2a + (n - 1)d] \qquad \text{II}'$$

In the arithmetic progression there are five quantities involved, viz., a, d, n, l, S. If any three of these are given, formulas I and II (or II′) can be used to find the other two.

EXERCISES

1. Using $S = \frac{n}{2}(a + l)$, find the sum of $1 + 2 + \dots + 10$. Note that $a = 1$, $d = 1$, $n = 10$. Check the sum by actually adding the numbers.
2. Find the sum of $1 + 2 + 3 + \dots + 100$.
3. Find the sum of all odd integers from 1 to 99.
4. Find the sum of all even integers from 2 to 100.

The sum of the results in exercises 3 and 4 should equal that found in exercise 2.

5. Find the sum of all even integers from 2 to 1000 and of all odd integers from 1 to 999. Check by finding the sum of all integers from 1 to 1000.

194. Summary of problems in A.P.—As noted on page 196, any two of the numbers a, d, n, l, S can be found when the other three are given. This gives rise to ten essentially different problems.

	Given	Find
1.	a,d,n	l,S
2.	a,d,l	n,S
3.	a,d,S	n,l
4.	a,n,l	d,S
5.	a,n,S	d,l
6.	a,l,S	d,n
7.	d,n,l	a,S
8.	d,n,S	a,l
9.	d,l,S	a,n
10.	n,l,S	a,d

If any three of the numbers a, d, n, l are given, the one remaining is found by using formula I (page 196). Then S is found by using formula II. This solves problems 1, 2, 4, and 7 above. In each of these there is only one answer. If S is one of the three given numbers, formula II or II′ must be used to find a fourth number. If n and l are not given, we have $S = \frac{n}{2}[2a + (n-1)d] = na + \frac{n^2}{2}d - \frac{n}{2}d$, which is a quadratic in n. This gives two values of n. Substituting in I, we have two values of l. If d, l, S are given, we have one linear and one second degree equation in the two unknowns a and n. This also gives two values of a and n.

EXERCISES

Verify the above by finding the remaining two numbers in each of the following. Check your answers.

1. $a = 3$, $d = 2$, $n = 7$
2. $a = 1$, $d = 1\frac{1}{2}$, $l = 15\frac{1}{2}$
3. $a = 2$, $d = 3$, $S = 40$
4. $a = 4$, $n = 7$, $l = 6$
5. $a = 2$, $n = 8$, $S = 34$
6. $a = -3$, $l = 15$, $S = 42$
7. $d = -2$, $n = 10$, $l = -6$
8. $d = -\frac{1}{3}$, $n = 12$, $S = 62$
9. $d = \frac{3}{4}$, $l = 13$, $S = 90$
10. $n = 16$, $l = -5/2$, $S = 20$

195. Arithmetic means.—Half the sum of two numbers is called the arithmetic mean between the two numbers. That is, $\frac{a+b}{2}$ is the arithmetic mean between a and b. The three numbers a, $\frac{a+b}{2}$, b are easily shown to be an arithmetic sequence. That is, $\frac{a+b}{2} - a = b - \frac{a+b}{2} = \frac{b-a}{2}$. Hence the common difference is $\frac{b-a}{2}$. In any arithmetic sequence the terms between the first and the last terms are also called arithmetic means. Thus in 2, 4, 6, 8, 10, 12, the terms 4, 6, 8, 10 are the four arithmetic means between 2 and 12.

$$\frac{a+b}{2} - a = \frac{b-a}{2}$$
$$b - \frac{a+b}{2} = \frac{b-a}{2}$$

Any given number of arithmetic means may always be inserted between any two given numbers. When there is only one arithmetic mean between two numbers, it is referred to as *the* arithmetic mean of these numbers. Thus $\frac{a+b}{2}$ is *the* arithmetic mean of a and b.

Example. Insert 7 arithmetic means between 2 and 5.

SOLUTION: The result will be an arithmetic sequence in which $a = 2$, $l = 5$, and $n = 9$. As at the right, find $d = 3/8$. Then the sequence is 2, $2\frac{3}{8}$, $2\frac{6}{8}$, $3\frac{1}{8}$, $3\frac{4}{8}$, $3\frac{7}{8}$, $4\frac{2}{8}$, $4\frac{5}{8}$, 5.

$$l = a + (n-1)d$$
$$5 = 2 + 8d$$
$$8d = 3$$
$$d = 3/8$$

EXERCISES

1. Find the 10th and 23d terms in 3, 8, 13, ...
2. Find the 97th and 276th terms in 21, 27, 33, ...
3. Find the 14th and 37th terms in 1, $1\frac{1}{2}$, 2, ...
4. Find the last term in 3, 9, 15, . . . if $n = 20$.
5. Find the last term in -1, -3, . . . if $n = 20$.
6. Find the sums of the A.P. whose terms are given in examples 4 and 5.
7. Find the sum of the A.P. whose terms are -4, -2, 0, . . . if $n = 28$.
8. Find the first term and the sum of a series of 14 terms in which $d = -1$, and $l = 7$.
9. Find d and S in a series in which $a = -7$, $n = 19$, and $l = 34$.

10. Insert 5 arithmetic means between -4 and 10.
11. Insert 6 arithmetic means between -8 and -1.
12. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.

In each of the following find the values of the two letters that are not given. If a fractional or negative value is found for n, this shows that the problem is impossible. This is also the case when imaginary values are found.

13. $\begin{cases} d = 4 \\ l = 38 \\ S = 192 \end{cases}$
14. $\begin{cases} S = 36 \\ n = 4 \\ l = 12 \end{cases}$
15. $\begin{cases} S = 105 \\ d = 4 \\ a = 3 \end{cases}$
16. $\begin{cases} d = -1 \\ l = -5\frac{1}{2} \\ n = 9 \end{cases}$
17. $\begin{cases} S = 20 \\ l = 13 \\ d = 3 \end{cases}$
18. $\begin{cases} d = 6 \\ n = 6 \\ S = 120 \end{cases}$
19. $\begin{cases} n = 5 \\ l = 21 \\ d = 6 \end{cases}$
20. $\begin{cases} S = -27 \\ a = 2\frac{1}{2} \\ l = -5\frac{1}{2}. \end{cases}$
21. $\begin{cases} a = 4 \\ n = 17 \\ S = 272 \end{cases}$
22. $\begin{cases} a = -9 \\ n = 7 \\ l = 16 \end{cases}$
23. $\begin{cases} a = -1 \\ l = 42 \\ S = 246 \end{cases}$
24. $\begin{cases} d = 3 \\ n = 17 \\ S = 68 \end{cases}$

PROBLEMS

1. Show that $1 + 3 + 5 + \ldots + n = k^2$, where k is the number of terms.
2. Show that $2 + 4 + 6 + \ldots + n = k^2 + k$, where k is the number of terms.
3. Find the sum of the $2k$ positive integers. Compare this result with the sum of the results in problems 1 and 2. Explain why this should be as you find it.
4. Find the sum of all even integers from 10 to 380 inclusive.
5. Find the sum of all odd integers from 11 to 379 inclusive.
6. Find the sum of all integers from 10 to 380 inclusive. How does this result compare with the sum of the results in problems 4 and 5? Explain why these results should be as you find them.
7. There are 16 rows of billiard balls in a symmetrical triangular arrangement on a table, with 46 balls in the first row and 3 less balls in each other row than in the one preceding it. How many balls are on the table?
8. The horizontal base of a right triangle is 15 feet long and the side perpendicular to the base is 45 feet long. At intervals of 1 foot on the base, a perpendicular is drawn to the base and reaches to the hypotenuse. Find the sum of the lengths of all perpendiculars, including the vertical leg of the triangle.
9. The 4th term of an A.P. is 215 and the 44th term is 55. Find the sum of the first 20 terms.
10. Find the sum of all positive integral multiples of 5 which are less than 498.

11. Find the sum of the first 38 positive integral multiples of 3.

12. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

13. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

14. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

15. If a clock strikes the hours only, how many times does it strike in one day?

16. A falling body drops 16.1 feet the first second, 3 times as far the next second, 5 times as far the third second, and so on; how far will it fall in t seconds?

17. Logs are piled so that the second layer contains one less than the first, the third one less than the second, and so on. If the first layer contains 15 logs, how many are there in a pile with 8 layers?

18. The sum of 5 numbers in arithmetic progression is 25 and the sum of their squares is 135; find the numbers.

19. In an arithmetic progression of 7 terms the sum is 140 and the product of the first and last term is 175; find the terms of the progression. Interpret the meaning of the two solutions.

20. In a progression of 9 terms the sum is 72 and the product of the fourth and the ninth terms is 84; find the terms of the progression. Interpret the meaning of the two solutions.

21. Find the terms of an arithmetic progression of 8 terms if the difference between the first and the last term is 42 and the last term is equal to the square of the first term. Interpret the meaning of the two solutions.

22. The 12th, 85th, and last terms of an A.P. are 38, 257, 395 respectively; find the number of terms.

23. Find the sum of all positive and negative integral multiples of 6 between -55 and 357.

24. How many terms must be taken of the series 42, 39, 36, . . . to make the sum equal to 312?

25. How many terms must be taken of the series $-6\frac{4}{5}, -6\frac{2}{5}, -6, \ldots$ to make the sum equal to $-52\frac{4}{5}$?

26. The 3rd and 7th terms of an A.P. are 7 and 19; find the 15th term. Also find the sum of the first 15 terms.

27. Insert 36 arithmetic means between $8\frac{1}{2}$ and $2\frac{1}{3}$. Find the sum of the resulting series.

28. Show that the formula $S = n/2\ [2a + (n - 1)d]$ may be translated into the following rule.

The sum of an arithmetic series is equal to the arithmetic mean of the first and the last terms multiplied by the number of terms.

196. Geometric sequence; geometric series.—A sequence of numbers of the type 1, 2, 4, 8, 16, . . . , in which each term is multiplied by the same number to form the succeeding term, is called a geometric sequence. In the sequence just given the multiplier, or ratio, is 2. In general, if a is the first term and r the fixed multiplier or ratio, then the terms of a geometric sequence are a, ar, ar^2, ar^3, . . . , ar^{n-1}.

The indicated sum of the terms in a geometric sequence is called a geometric progression, or a geometric series, which may be denoted by G.P. The standard form of a geometric series is:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Note that the nth term of this progression is ar^{n-1} where n is the number of terms. That is, in each term the exponent of r is one less than the order of the term. In the 2nd term the exponent is 1, in the 3rd term it is 2, in the 4th term it is 3, and in general, in the nth term the exponent is $n - 1$. If l denotes the last term, then $l = ar^{n-1}$, where a is the first term, r the ratio, and n the number of terms. Note that any term divided by the one preceding it gives the common ratio, or r.

$$l = ar^{n-1}$$

Example. Find the 7th term in the geometric sequence ½, ⅔.

SOLUTION: $a = \frac{1}{2}$, $ar = \frac{2}{3}$. Therefore $r = \frac{2}{3} \div \frac{1}{2} = 4/3$. Hence the 7th term $= ar^6 = \frac{1}{2}\left(\frac{4}{3}\right)^6 = \frac{2048}{729}$.

$$ar \div a = \frac{2}{3} \div \frac{1}{2} = \frac{4}{3} = r$$
$$ar^6 = \frac{1}{2}\left(\frac{4}{3}\right)^6 = \frac{2048}{729}$$

EXERCISES

In each of the following the first two terms of a geometric sequence, and also the number of terms are given. Find the ratio and the last term.

1. 1, 2, . . . to 8 terms.
2. ½, 1, . . . to 7 terms.
3. 1, ½, . . . to 6 terms.
4. 2, −3/2, . . . to 5 terms.
5. ⅔, −½, . . . to 9 terms.
6. −⅔, ¾, . . . to 5 terms.
7. 5, 4, . . . to 6 terms.
8. 8, 4, . . . to 16 terms.

197. Sum of a G.P.—A compact formula for the sum of a G.P. may be found as follows. Let the sum be S. Then,

$$S = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \qquad (1)$$
$$Sr = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \qquad (2)$$

The second equation is obtained by multiplying (1) by r. Subtracting, we have $S - Sr = a - ar^n$. Dividing by $1 - r$ gives formula II at the right. If the signs are changed in both numerator and denominator, we have formula II′. Formula I is repeated here for convenience. If r is positive and less than 1, formula II′ may be used; otherwise it is best to use formula II.

$$l = ar^{n-1} \qquad \text{I}$$
$$S = \frac{a - ar^n}{1 - r} \qquad \text{II}$$
$$S = \frac{ar^n - a}{r - 1} = \frac{a(r^n - 1)}{r - 1} \qquad \text{II}'$$

Study the proof above by writing out completely a series of eight terms, then multiplying by r and subtracting. Note how the terms cancel.

Problem. Find the sum of a G.P. in which $a = \frac{2}{3}$, $r = 2$, $n = 5$. Note that in the statement at the right formula II′ is used (not II).

$$S = \frac{a[r^n - 1]}{r - 1} = \frac{\frac{2}{3}[2^5 - 1]}{2 - 1} = \frac{62}{3}$$

EXERCISES

Find the sum of the geometric series if in each case the first two terms and the number of terms are given.

1. 1, 2, . . . to 10 terms.
2. 1, ½, . . . to 10 terms.
3. 9, .09, . . . to 3 terms.
4. 1, 1.05, . . . to 5 terms.
5. 1, 1.1, . . . to 5 terms.
6. In the series 1, 2, 4, find the difference between the 10th term and the sum of the first 9 terms.
7. Using $S = \dfrac{a(r^n - 1)}{r - 1}$, verify II′ by factoring $r^n - 1$ and canceling $r - 1$.
8. There is a story that an Arab had a beautiful steed that he had always refused to sell. When the king asked him the price of the horse he said: "This steed will not be sold to anyone but the king; but the king may have him if he gives me one grain of wheat for the first nail in the horse's shoes, 2 grains for the second nail, 4 grains for the third nail, 8 grains for the fourth, and so on up to the last of the 32 nails in the horse's shoes." How would you find the number of grains of wheat asked as the price of the horse?

198. *Geometric means.*—In a geometric sequence the terms between the first and the last terms are called geometric means between these terms. Thus in 1, 2, 4, 8, 16, 32, 64, the terms 2, 4, 8, 16, 32 are the five geometric means between 1 and 64.

If there is only one geometric mean between two numbers, then this mean is called *the* geometric mean of these numbers. In a, ar, ar^2, ar is the geometric mean between a and ar^2. If $ar^2 = b$, then $r = \sqrt{\frac{b}{a}}$ and $ar = a\sqrt{\frac{b}{a}} = \sqrt{ab}$. Hence $\sqrt{ab}$ is the geometric mean of a and b.

If in a series of n terms the first term is a and the last term is l, then $l = ar^{n-1}$, or $r = \left(\frac{l}{a}\right)^{1/(n-1)}$. Hence the geometric means are $a\left(\frac{l}{a}\right)^{1/(n-1)}$, $a\left(\frac{l}{a}\right)^{2/(n-1)}$, $a\left(\frac{l}{a}\right)^{3/(n-1)}$, $\ldots$, $a\left(\frac{l}{a}\right)^{(n-2)/(n-1)}$. Clearly the means between a and $l = ar^{n-1}$ are ar, ar^2, $\ldots$, ar^{n-2}, where r has the value $\left(\frac{l}{a}\right)^{1/(n-1)}$. Except in special cases these terms involve roots that can only be indicated or approximated.

Example 1. Insert 6 geometric means between 1 and 128.

SOLUTION: The geometric sequence will have 8 terms. Hence $r = \left(\frac{128}{1}\right)^{1/7} = \sqrt[7]{128} = 2$, and the required means are 2, 4, 8, 16, 32, 64.

Example 2. Insert 5 geometric means between 2 and 8.

SOLUTION: The geometric sequence will have 7 terms. Hence, $r = \left(\frac{8}{2}\right)^{1/6} = 4^{1/6} = 2^{1/3}$, and the required means are $2 \cdot 2^{1/3}$, $2 \cdot 2^{2/3}$, $2 \cdot 2^1$, $2 \cdot 2^{4/3}$, $2 \cdot 2^{5/3}$; or $2\sqrt[3]{2}$, $2\sqrt[3]{4}$, 4, $4\sqrt[3]{2}$, $4\sqrt[3]{2^2}$.

EXERCISES

1. Insert 5 geometric means between 1 and 2.
2. Insert 7 geometric means between 1 and 16.
3. Insert 9 geometric means between 1 and 1024.
4. Insert 4 geometric means between ½ and ⅔.
5. Insert 3 geometric means between ⅔ and 3/2.
6. Insert 5 geometric means between ¾ and 9/4.
7. Insert n geometric means between a and b. Verify the solution by showing that the $(n + 2)d$ term in the sequence is b.

199. Summary of problems in G.P.—As in the case of A.P., there are involved five quantities, a, r, n, l, and S, the common ratio r replacing the common difference d. Then exactly as on page 197 there are ten problems. In practical applications by far the most important of these are the ones directly represented by formulas I and II. Some of the eight remaining problems are a little complicated and for practical purposes are of no great importance. Problems that might lead to these are usually solved in other ways. In some of the problems that follow such other ways are pointed out. One important case—namely, finding r when a, n, and l are given—has been solved in a preliminary way on page 203.

200. Harmonic sequence.—If the numbers $a_1, a_2, a_3, \ldots, a_n$ form an arithmetic sequence, then the reciprocals of these numbers are said to form a harmonic sequence. Thus $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots 1/n$ is a harmonic sequence, since $1, 2, 3, 4, \ldots, n$ is an arithmetic sequence. In general,

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots, \frac{1}{a+(n-1)d}$$

is a harmonic sequence.

The name "harmonic" for this sequence is due to the following fact. If strings of the same weight and tension have lengths in proportion to $1, \frac{1}{2}, \frac{1}{3}, \ldots$, then if any two of these strings are set in vibration, they will produce harmonious sounds.

EXERCISES

1. Write three additional terms in the harmonic sequence $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}$. What is the twelfth term in this sequence?

2. Decide whether or not each of the following is a harmonic sequence.

(a) $\frac{3}{2}, \frac{6}{7}, \frac{3}{5}, \frac{6}{11}, \frac{3}{8}$

(b) $\frac{1}{5}, \frac{3}{14}, \frac{3}{13}, \frac{1}{4}, \frac{3}{11}$

(c) $\frac{4}{13}, \frac{2}{7}, \frac{4}{15}, \frac{1}{4}, \frac{4}{17}$

(d) $\frac{8}{3}, -8, -\frac{8}{5}, -\frac{8}{9}, -\frac{11}{8}$

(e) $2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}$

(f) $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}$

(g) $1, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{17}$

(h) $\frac{2}{3}, \frac{6}{7}, \frac{6}{11}, \frac{3}{2}, \frac{6}{13}$

201. ***Harmonic means.***—When three quantities form a harmonic sequence, the middle one is *the* harmonic mean between the other two. If H is the harmonic mean between a and b, then $1/a$, $1/H$, $1/b$ form an arithmetic sequence. That is, $1/H - 1/a = 1/b - 1/H$. Hence H is found as at the right. Therefore $2ab/(a+b)$ is the harmonic mean between a and b. Check the computation at the right.

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \quad (1)$$

$$\therefore \quad \frac{2}{H} = \frac{1}{b} + \frac{1}{a} \quad (2)$$

$$\text{and} \quad H = \frac{2ab}{a+b} \quad (3)$$

202. ***Relation between the arithmetic, geometric, and harmonic means.***—Let these means be represented by A, G, and H. Then these means between a and b are

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}$$

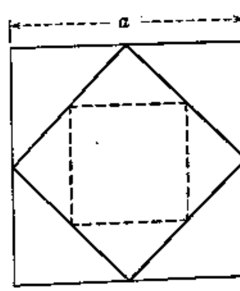
$$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$$

Since $AH = G^2$ (see at the right), it follows that the geometric mean between a and b is the geometric mean between the arithmetic and the harmonic means.

PROBLEMS

1. The second term of a geometric progression is 2 and the seventh term is 64; find the first seven terms of the progression.

2. The third term of a geometric progression is 1, and the sixth term is ⅛; find the sum of the first seven terms.



3. The sides of a square are a. A square is constructed having its vertices in the middle points of the sides of the first square; a third square is constructed having this same relation to the second square, and so on; what is the length of the sides of the sixth square? What is the sum of the areas of these six squares?

4. Certain bacteria under favorable conditions will double their number in 3 hours; by how much will the original number be multiplied in 24 hours?

5. An air pump removes one-fourth of the air in the receiver at the first stroke, one-fourth of what remains at the second stroke, and so on; what fraction of the original amount will remain after the fourth stroke?

6. Which is greater, the arithmetic mean or the geometric mean (positive) between two positive numbers?

7. The product of three consecutive terms of a geometric progression is 1000. Find the second of these terms.

8. Four numbers are in geometric progression. The sum of the second and third of these numbers is 18, and the sum of the first and fourth is 27. Find the numbers.

9. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third the sums will be in geometric progression. Find the numbers.

10. For what values of k do the three quantities $(k + 3)$, $(6k + 3)$, and $(20k + 3)$ form a G.P.? Can you find a value of k for which these numbers form an A.P.?

11. An investment paid a man, each year after the first, twice as much as in the preceding year. If his investment paid him $13,500 in the first four years, how much did it pay in the first and the fourth years? How much did it pay in the second and third years?

12. In a lottery it is agreed that the first ticket drawn will pay its owner $10 and each succeeding ticket twice as much as the preceding one. Find the total amount paid on the first 10 tickets drawn.

13. The radiator of a motor truck contains 10 gallons of water. We draw off 1 gallon and replace it with alcohol; then we draw off 1 gallon of the mixture and replace it by alcohol, etc., until 9 drawings and replacements have been made. How much alcohol is in the final mixture?

14. If the corresponding terms of two geometric progressions are multiplied, do the products form a geometric progression? That is, $a, b, c, d, \ldots$ and $p, q, r, s, \ldots$ are geometric progressions. Does it follow that ap, bq, cr, ds, . . . is a geometric progression? Prove.

15. If in problem 14 the sums of the corresponding terms are taken instead of the products, is the result an arithmetic series?

16. In 1933 there were 1000 students enrolled in a certain college. If the attendance each succeeding year is 5 per cent more than it was for the preceding year, what was the enrollment in 1936? Find answer to the nearest unit.

SUGGESTION: Show that $1000(1.05)^3$ represents the desired answer.

17. The arithmetic mean of two numbers is 30 and their geometric mean is 18. What are the numbers?

18. Find the arithmetic, harmonic, and geometric means of 4 and 3/2; of a and $a + x$.

19. Insert three geometric means between 3 and 48.

20. Insert three harmonic means between ½ and 1/10.

21. If G is the geometric mean between two quantities A and B, show that the ratio of the arithmetic and harmonic means of A and G is equal to the ratio of the arithmetic and harmonic means of G and B.

MISCELLANEOUS PROBLEMS INVOLVING PROGRESSIONS

1. If \$500 is to be divided among 10 men so that the first one receives \$5 and each succeeding man obtains a fixed amount more than the preceding man, how much will the 10th man receive?

2. At a bazaar, tickets are marked with the consecutive even integers 2, 4, 6, . . . and are drawn at random by those entering. If each person pays as many cents as the number on his ticket, how much money is received if 1000 tickets are sold?

3. A man piles 152 logs in layers so that the top layer contains 2 logs and each lower layer has one more log than the layer above. How many logs will be in the lowest layer?

4. A man piles 68 logs, putting 12 in the lowest layer, 11 in the next layer and so on. How many will he have left for the top layer?

5. In a professional golf tournament, the total prize money of \$5187 is divided among the six players with lowest scores, so that each man above the lowest receives $\frac{2}{5}$ as much as the man next below him. How much does the man with the lowest score receive?

6. A body dropped from a position of rest in a vacuum near the earth's surface will fall approximately 32 feet farther in each second, after the first, than in the preceding second. If a body falls 10,000 feet in 25 seconds, how far does it fall in the first second?

7. In creating a vacuum in a container, a pump draws out $\frac{1}{4}$ of the remaining air at each stroke. What part of the original air has been removed by the end of the 7th stroke?

8. If two-fifths of the air in a container is removed by each stroke of an air pump, what fraction of the air has not been removed after five strokes of the pump?

9. If \$570 is to be divided among ten persons so that each after the first one receives \$10 more than the preceding one, how much does the last one receive?

10. A vessel containing milk was emptied of one-third of its contents and then filled with water. This was done four times. What portion of the original contents was then in the vessel?

11. A debt of \$18,000 is to be paid in 40 annual installments that form an arithmetic series. When 30 installments have been paid \$6,000 still remains to be paid. What were the amounts of the last payments?

12. The three numbers a, b, c, form an arithmetic sequence. If $a + b + c = 60$ and if $a + 1, b, c$ form a geometric sequence, find the values of a, b, c.

13. Prove that A is the arithmetic mean between $A + \sqrt{(A + G)(A - G)}$ and $A - \sqrt{(A + G)(A - G)}$, and that G is the geometric mean between these quantities.

203. Investments.—A fairly extensive "mathematical theory of investments" has been developed, and courses devoted exclusively to it are now given in our colleges and universities. The central element in this theory is the geometric progression by means of which problems involving compound interest are handled. While in commercial practice interest is commonly paid at the end of each interest period, a large body of important practical problems involve compound interest. In this book we shall study the main fundamentals of the theory of investments, beginning with the construction of a compound interest table.

204. The compound interest table.—As in elementary arithmetic, to find the amount (interest plus principal) of an investment at the end of one year, multiply the investment by $1 + i$, i being the rate of interest expressed as a decimal. A certain amount of money, called the principal, is invested at 3% interest, compounded annually. To make a compound interest table for this rate let the investment be $1.

At the end of the first year the amount is $1.03 \times \$1 = \$1.03$, at the end of the second year it is $1.03 \times \$1.03 = (\$1.03)^2$, at the end of the third year $1.03 \times (\$1.03)^2 = (\$1.03)^3$, and so on. In the table at the right each term is obtained by multiplying the preceding term by 1.03. We therefore have a geometric sequence in which $a = 1.03$ and $r = 1.03$.

Terms		Amt. at 3%
1.		= 1.0300 0000
2.	$(1.03)^2$	= 1.0609 0000
3.	$(1.03)^3$	= 1.0927 2700
4.	$(1.03)^4$	= 1.1255 0881
5.	$(1.03)^5$	= 1.1592 7407
6.	$(1.03)^6$	= 1.1940 5230
7.	$(1.03)^7$	= 1.2298 7387
8.	$(1.03)^8$	= 1.2667 7008
9.	$(1.03)^9$	= 1.3047 7318
10.	$(1.03)^{10}$	= 1.3439 1638

The reason for carrying the result to a large number of decimal places is that the principal on which interest is to be computed may be large and that the final result must be correct to the nearest cent. The simplest way to construct such a table is actually to multiply successively by 1.03. In practice, books of such tables are used which run to 100 terms or even more.

On page 415 of this book a brief 4-place compound interest table is given for use in practice. When the investment in question is fairly large this table will not give the result very closely.

205. *The interest period.*—Interest is always compounded at specified equal intervals. Thus, it may be compounded annually (once each year), semi-annually (every six months), quarterly, and even monthly. The time interval at the *end* of which interest is compounded is called the interest period or interest term. The rate given is the annual rate. If the rate is 4% compounded semi-annually, then the rate for the six months is 2%. That is, at the end of every six months interest at 2% of the principal drawing interest is computed and added to the principal. This is done by multiplying the principal by 1.02. If the annual rate is 4% compounded quarterly, then the principal is multiplied by 1.01, the rate used being 1%. If, in general, the annual rate is $i\%$ and the number of interest periods in one year is m, then the rate used for each interest period is $i/m\%$, this rate being usually denoted by j. Then the amount of \$1 for one year, compounded m times a year, is $(1 + i/m)^m$ or $(1 + j)^m$. Hence under these conditions, the amount in n years is $(1 + i/m)^{mn}$, or $(1 + j)^{mn}$. Note that in the formula $(1 + i)^n$, the expression $1 + i$ is now replaced by $(1 + j)^m$.

EXERCISES

1. If \$1250 carries 3% interest compounded annually, what is the amount at the end of 7 years?

SUGGESTION: Use the table on page 208. Multiply 1.22987387 by 1250.

How many decimals in the table are needed to make the result correct to the nearest cent? How many decimals would be needed if the investment were \$125,000?

2. Construct a table like the one on page 208, using 1% as the rate.

3. A savings bank pays interest at 2% compounded semi-annually. If \$895.00 is deposited in this bank, what will be the amount at the end of 5 years? Use the table constructed under example 2. Note that there will be ten interest periods.

4. If you did not have the table, how would you solve example 3? How would this work compare with that required to construct the table?

5. From $(1.03)^{20} = (1.03)^{10} \cdot (1.03)^{10}$ find the amount of \$1 at 3% for 20 years. Find $(1.03)^{10}$ on page 208.

6. From the table on page 208 find $(1.03)^{15}$ by one multiplication.

7. If you had a table giving compound amounts for 50 terms, how would you find the amount for 90 terms?

8. State a rule for finding the compound amount for any number of years if you have a table for 50 terms.

206. Annuities.—A fairly common business arrangement leads to what is called an annuity. Examples will illustrate: A man takes out an insurance policy and pays a stated premium at the beginning of each year. That is, he pays the company an annuity for the term of years during which the payments continue. Again, an insurance company agrees to pay a certain amount at the end of each year for a certain number of years, or possibly during the life of an individual. This is also an annuity.

In general, an annuity is a series of equal payments made at the beginning or end of equal periods of time. In this book we shall deal only with the case when the payments begin at the end of the first period. This kind of annuity is called an ordinary annuity, and it is for this kind of annuity that formulas are given on the following pages. To apply these formulas to the case of premiums paid on an insurance policy, we regard the first period as beginning one year before the first premium is paid (in case premiums are paid yearly). The case given below constitutes a ten-term annuity, the last payment being made nine years after the first payment.

207. Problems on annuities.—There are two main problems in connection with annuities:

1. What will be the accumulated value, or amount of an annuity, in a given time?

2. What is the present value, or cost, of a given annuity?

In both problems, investment at compound interest is assumed. The insurance company invests the premiums as received, and figures a certain rate per cent compound interest on all such investments. Three per cent is now a usual rate. Suppose the first premium is invested for 9 years, the second for 8 years, and so on. For the purpose of computing let the premium be \$1. Then the moment the last payment is made this payment will be worth \$1. The payment made one year earlier will be worth \$1.03, the payment made two years earlier will be worth $(1.03)^2$, and so on. Hence the total value of ten payments will be

$$1 + 1.03 + (1.03)^2 + (1.03)^3 + \dots + (1.03)^9,$$

which is a geometric series with $a = 1$, $r = 1.03$, and $n = 10$.

208. The annuity table.—By evaluating the series for $n = 2, 3, \ldots, 10$ we may construct a table as at the right.

Payments	Value
1.	1.0000 0000
2.	2.0300 0000
3.	3.0909 0000
4.	4.1836 2700
5.	5.3091 3581
6.	6.4684 0988
7.	7.6624 6218
8.	8.8923 3605
9.	10.1591 0613
10.	11.4638 7931

But the terms of this progression are evaluated in the table on page 208. Hence if we add \$1 and \$1.03 we get the second line in the table at the right. If to this we add $(1.03)^2$ as given on page 208, we have the third line at the right, and so on. Hence the table on page 208 is fundamental in constructing the annuity table, and also, as we shall see later, in constructing other tables. See page 212.

EXERCISES

1. Extend the table on page 208 to 16 terms. Using your result, find the amount in 16 years of \$3560 invested at 3% compounded annually.

SUGGESTION: This work can be done quickly as shown in the form at the right. The first line gives the amount for 10 terms. This is multiplied by 3, the product put two places to the right, and added to the first line. This gives $(1.03)^{11}$. Explain how $(1.03)^{12}$ is obtained.

$$\begin{aligned}(1.03)^{10} &= 1.34391638\\ &403174914\\ \hline (1.03)^{11} &= 1.38423387\\ &415270161\\ \hline (1.03)^{12} &= 1.42576089\end{aligned}$$

2. Using the above table, find the amount of an annuity of \$100 for 8 years.

3. Extend the above table to 16 years. Use the table made in exercise 1. With this table, find the amount of an annuity of \$785 running 16 years at 3% compounded annually.

4. Use the table constructed in exercise 2, page 209, to construct an amount of annuity table for 10 payments with rate 1%.

5. Using the table constructed under exercise 4, find the amount of an annuity of \$275 for 6 years.

6. What is the amount of an annuity of \$276 for 6 years if the rate of interest is 3%? Use the table in §208. What is the reason for the difference between the amounts found in exercises 5 and 6?

7. Using the formula $S = \dfrac{a(r^n - 1)}{r - 1}$, find a formula for the sum of the geometric series $1 + 1.03 + (1.03)^2 + \ldots + (1.03)^9$. See the bottom of page 210.

8. Using the value of $(1.03)^{10}$ given on page 208, find correct to the nearest cent the value of the sum found in exercise 7.

209. Computing amount of annuity using the compound interest table.—Let i be the rate of interest expressed as a decimal. Then the amount of an investment of p dollars at the end of one year is $p(1+i)$. If $1 is used as the original investment (principal), the amounts at the end of successive years will be

$$(1+i), (1+i)^2, (1+i)^3, \ldots, (1+i)^n$$

The accumulated value of an annuity of n yearly payments of $1, as studied on page 211, will be

$$A = 1 + (1+i) + (1+i)^2 + (1+i)^3 + \ldots + (1+i)^{n-1}$$

This is a geometric progression of n terms in which $a = 1$ and $r = 1 + i$. Hence the sum (see page 202) is

$$\frac{a(r^n - 1)}{r - 1} = \frac{1[(1+i)^n - 1]}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}$$

(See exercise 7, page 211.)

Let $i = .03$ and $n = 10$. Then from the table on page 208 $(1.03)^{10} =$ 1.3439 1638. Hence $\frac{(1.03)^{10} - 1}{.03} = \frac{.34391638}{.03} = 11.4638793$, which checks to seven decimals with the value found on page 211. If we have a table of the values of $(1.03)^n$ correct to eight places of decimals, we can write the amount of an annuity for any number of years for $i = .03$ by subtracting 1 and dividing by .03. The result will certainly be correct to six places of decimals, and very likely to seven. Both steps may be done at sight and only the result written down.

Hence a table of $(1.03)^n$ enables us to write at sight a table of $\frac{(1.03)^n - 1}{.03}$, which is a table giving amounts of an annuity. In practice we can write the amount for any desired number of years without writing a complete table. On page 213 is a table giving the values of $(1.03)^n$ from $n = 1$ to $n = 50$. That is, this table gives the amount of $1 invested at 3% compounded annually for any number of years from 1 to 50.

For practical use tables are provided giving such values up to $n = 200$ for thirty or more of the different rates that are in practical use. Similar tables are such as those begun on pages 211 and 215.

EXERCISES

1. With the table at the right, construct a table giving the amount of an annuity of $1, using 3% compounded annually as the rate of interest. Use the method suggested on page 212.

2. Find the amount of $100 invested yearly at 3% compounded annually for 17 years.

3. I buy a vacant lot for $650 for which I pay cash. At the end of each year I pay $22.50 in taxes on the lot. If I figure interest at 3% compounded annually on all the money invested, how much does this lot cost me at the end of 7 years?

4. How much does this lot cost me, figuring interest compounded annually at 4%? at 5%? at 6%? Use the tables on pages 415, 416.

5. A young man spends $1150 a year for 4 years going to college. At 6% interest compounded annually, how much is invested in his college education at the end of the fourth year? Assume that each year's expense is paid out at the beginning of the year.

6. State in detail how you would construct a table like the one at the right, using 4% as the rate.

7. The last 15 years I have paid $15.50 each year in taxes on a vacant lot. If I figure interest at 3% on each payment compounded annually, how much do these taxes amount to at the end of the present year? Assume that the tax has been paid at the end of each year.

8. As a speculation I buy a timber tract for $1000. If I figure that this investment should pay interest at 6% compounded semi-annually, how much must this lot be worth at the end of 20 years? Note that I regard this rate of interest to be sufficient to cover yearly taxes besides giving a normal return on my investment. Do I figure the normal return on my investment to be more or less than 6%?

9. At 3% interest compounded yearly, in about how many years will an investment double itself?

10. At 3% interest compounded yearly, in about how many years will an investment multiply itself by 3? by 4?

Comp. Int. Years	$1 at 3% $(1+i)^n$
1	1.0300 0000
2	1.0609 0000
3	1.0927 2700
4	1.1255 0881
5	1.1592 7407
6	1.1940 5230
7	1.2298 7387
8	1.2667 7008
9	1.3047 7318
10	1.3439 1638
11	1.3842 3387
12	1.4257 6089
13	1.4685 3371
14	1.5125 8972
15	1.5579 6742
16	1.6047 0644
17	1.6528 4763
18	1.7024 3306
19	1.7535 0605
20	1.8061 1123
21	[illegible]
22	1.9161 0341
23	1.9735 8651
24	2.0327 9411
25	2.0937 7793
26	2.1565 9127
27	2.2212 8901
28	2.2879 2768
29	2.3565 6551
30	2.4272 6247
31	2.5000 8035
32	2.5750 8276
33	2.6523 3524
34	2.7319 0530
35	2.8138 6245
36	2.8982 7833
37	2.9852 2668
38	3.0747 8348
39	3.1670 2698
40	3.2620 3779
41	3.3598 9893
42	3.4606 9589
43	3.5645 1677
44	3.6714 5227
45	3.7815 9584
46	3.8950 4372
47	4.0118 9503
48	4.1322 5188
49	4.2562 1944
50	4.3839 0602
—	—
n	

210. The present value of a future income.—A debt of \$100 due one year from now has a present value such that if invested now, the amount in one year will be \$100. If the present value is d (discounted value) and the rate is 3%, then $d(1.03) =$ \$100. If extended to six decimal places the present value is \$97.087379.

$$d(1.03) = \$100$$
$$d = \frac{\$100}{1.03} = \$97.087379$$

If the debt is due in two years, then using compound interest we find the present value is $100/(1.03)^2$; if due in three years the present value is $100/(1.03)^3$, and so on. If the future debt is \$1 and the rate of interest is i compounded annually, the present values for the debt due in one year, two years, . . . , up to n years are

$$\frac{1}{1+i}, \frac{1}{(1+i)^2}, \frac{1}{(1+i)^3}, \frac{1}{(1+i)^4}, \ldots, \frac{1}{(1+i)^n}$$

respectively, the rate i being written as a decimal. Clearly a table of present values can be made by dividing 1 in succession by the values of $1+i$, $(1+i)^2$, . . . as given in the table on page 415. When the rate is 3% the divisors are those given on page 213. This table is given on page 215; other rates of interest are shown on page 421.

The division required in constructing such a table by this method is laborious when carried out "longhand"; it is best done by using logarithms. (See Chapter 14.)

211. The present value of an annuity.—Clearly the present value (discounted value) of an annuity is the sum of the present values of the individual payments, namely,

$$D = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n}$$

This is a geometric series of n terms with $a = \frac{1}{1+i}$, and $r = \frac{1}{1+i}$.

Hence $$D = \frac{\frac{1}{1+i}\left[1 - \left(\frac{1}{1+i}\right)^n\right]}{1 - \frac{1}{1+i}} = \frac{1 - \left(\frac{1}{1+i}\right)^n}{i}.$$

Using n negative exponents, we have,

$$\left(\frac{1}{1+i}\right)^n = (1+i)^{-n},$$

and hence the formula is

$$D = \frac{1-(1+i)^{-n}}{i}.$$

Note that A (amount) is used for the accumulated value of the annuity, while D (discounted value) is used for the present value of the annuity.

To find the present value of an annuity of \$100 to run for 20 years, the rate being 3%, we therefore proceed as follows.

STEP 1. From 1 subtract .5537 . . ., the number opposite 20 in the table at the right. The result is .4463. . . .

STEP 2. Divide by .03, giving \$14.877. . . . Hence \$1487.7 . . . is the present value. For practical use, tables are constructed giving the present values for different rates. With tables such as the one at the right, the above is the most direct method for constructing tables giving the present values of annuities.

Present Value of \$1, Interest at 3% Compounded Annually

1	0.9708 7379
2	0.9425 9591
3	0.9151 4166
4	0.8884 8705
5	0.8626 0878
6	0.8374 8426
7	0.8130 9151
8	0.7894 0923
9	0.7664 1673
10	0.7440 9391
11	0.7224 2128
12	0.7013 7988
13	0.6809 5134
14	0.6611 1781
15	0.6418 6195
16	0.6231 6694
17	0.6050 1645
18	0.5873 9461
19	0.5702 8603
20	0.5536 7575
21	0.5375 4928
22	0.5218 9250
23	[illegible]
24	0.4919 3374
25	0.4776 0557
26	0.4636 9473
27	0.4501 8906
28	0.4370 7675
29	0.4243 4636
30	0.4119 8676
31	0.3999 8715
32	0.3883 3703
33	0.3770 2625
34	0.3660 4490
35	0.3553 8340
36	0.3450 3243
37	0.3349 8294
38	0.3252 2615
39	0.3157 5355
40	0.3065 5684
41	0.2976 2800
42	0.2889 5922
43	0.2805 4294
44	0.2723 7178
45	0.2644 3862
46	0.2567 3653
47	0.2492 5876
48	0.2419 9880
49	0.2349 5029
50	0.2281 0708

EXERCISES

1. Using the table at the right, find the present value of an annuity of \$250 yearly for 30 years if the rate is 3%.

2. Using the table on page 417, find the present value of the annuity in exercise 1 if the rate is 5%.

3. A property is bought for \$2500 cash and \$500 to be paid at the end of each year for 40 years. What is the total present value of this property, interest on the deferred payments being figured at 3%?

4. A man owes a debt of \$8000 due now. How much must he pay at the end of each year if he is to discharge this debt by 20 equal payments, the rate being 3%?

PROBLEMS

In solving the problems on this page, use the tables on pages 415–421.

1. A deposit of \$50 is made at the end of every six months in a bank paying 2% interest compounded semi-annually. What is the amount accumulated after 8 years? Note that this is equivalent to 16 years at 1% compounded annually. There are 16 payments.

2. What is the accumulated amount in 8 years if deposits of \$25 are made every three months in a savings bank paying 4% interest compounded quarterly?

3. In problem 2, what would be the accumulated amount if the interest were compounded yearly? Assume that \$100 is deposited at the end of each year. In this case what is the difference in the amounts?

4. Ten years ago I bought a vacant lot for \$1200. The taxes have been \$37.50 paid at the end of each year. Six years ago I paid \$800 on a sewer. If interest is figured at 4% compounded annually, what is my present investment in this lot?

SUGGESTION: Find (a) the accumulated amount of the buying price; (b) the accumulated value of the tax payments; (c) the accumulated amount of the cost of the sewer.

5. An annuity to run 15 years is bought for one cash payment. What is the cost if each payment of the annuity is \$2500, the rate of interest being figured at 3%? What is the cost of this annuity if interest is figured at 2%? at 4%? at 5%? As interest rates go down, what will be the effect on the cost of annuity? What would be the cost of this annuity if the rate were 1%?

6. A philanthropist wishes to give a college an amount sufficient to pay a professor's yearly salary of \$6500 at the end of each year for 25 years. How much must he give if interest is figured at 4%?

7. At present I wish to invest an amount which will provide an annuity of \$1200 to run for 20 years. If interest is 3% compounded annually, how much must I invest? How much would I have to invest if interest were 6%? How do you account for the difference between these amounts?

8. How much must I invest annually for 7 years so that the amount at the end of that time will be \$10,000? The interest is 3% compounded annually. What would be the yearly investment if the interest rate were 6%?

9. A man buys a house and agrees to pay \$4000 cash and \$1500 at the end of each year for 4 years. If money is worth 6%, what is the cash equivalent of the cost of the house?

10. A farm was bought for \$8000, \$2000 being paid in cash and the remainder in 12 equal semi-annual payments. What is the amount of each payment, interest on the unpaid part being figured at 6% compounded semi-annually?

11. A man wishes to deposit with a trust company a sum of money which will enable the trust company to pay his family \$2500 at the end of each year for 20 years. How much must the deposit be if interest is at 4% per annum?

12. A corporation wishes to set aside annually a sum of money which together with accumulated interest will be just sufficient to meet a payment of \$100,000 due in 5 years. How much must be set aside each year if interest is 4% per annum?

13. On December 1, 1930, a man purchased an annuity which enabled him to receive 15 annual payments each of \$500. What should he pay for the annuity if interest is computed at 3%?

14. A man saves \$500 each year and deposits his savings at the end of each year for ten years in a bank which pays 1½% compounded annually; he leaves the whole amount for five years after the final deposit has been made. What sum has he to his credit?

15. How much must I place on deposit at the end of each year at 3% compounded annually so as to be able to replace at the end of twelve years a machine that costs \$12,000?

16. A man makes a \$2000 cash payment on a house and agrees to pay \$1000 at the end of each of the next ten years. Figuring money worth 6% compounded annually, find the equivalent cash price of the house.

17. A father directed in a will that a certain son should be paid \$1200 at the end of each year for eight years. What amount set aside from the estate and placed at interest at 4% compounded annually will just meet these payments?

18. To meet the cost of replacing in eight years a machine which costs \$1200, a man makes equal annual deposits at the ends of the years. If he receives 3% compounded annually, find his annual deposit.

19. An investment will yield \$500 at the end of each year for 10 years. If money is worth 6% compounded annually, find the present value of the investment.

20. To create a fund of \$5000 at the end of 15 years, what equal payments should a man deposit at the end of each year in a savings account which earns 3%, compounded annually?

21. What equal payments should be made at the end of each year for 10 years to discharge a loan of \$8000, if money is worth 6%, compounded annually?

22. If I deposit \$300 at the end of each year for six years in a bank that pays 1½% compounded annually, what amount shall I have to my credit immediately after the sixth deposit?

23. The U. S. Postal savings can be made to pay 2% interest compounded annually by withdrawing the interest at the end of each year as it falls due and then redepositing it. Under this arrangement, what would be the amount in the preceding problem?

24. A man wills \$20,000 to a son, to be paid to him, principal and interest, in equal amounts at the end of each year for the following ten years. If money is worth 3% compounded annually, find the annual payment.

25. A college fraternity plans the discharge of a debt of \$20,000 in 10 years by a series of 10 equal annual payments to be made at the end of each year. If the unpaid balance of the debt bears interest at the rate of 5%, what amount must the fraternity pay each year?

26. If in the year 1600 a sum of \$2000 had been left to accumulate for 300 years, find its amount in the year 1900, reckoning compound interest at 4 per cent per annum.

27. What is the present value of an annuity of \$2000 due in 30 years allowing interest at 5 per cent per annum?

28. It is estimated that the possession of \$10,000 at the age of sixty-five will provide for old age. What equal annual savings for forty years (the final payment to be made at the age of sixty-five) will amount to this sum at 4% compounded annually?

29. If it is estimated that a man aged 65 will live 13 years, what annuity can he purchase for \$10,000, the rate of interest being 4%?

30. A man contracts to pay off a \$20,000 mortgage, both principal and interest, at 6% compounded annually, by equal annual payments at the end of consecutive years for eight years. Find his annual payment.

31. Find the present value of an investment which will yield \$200 at the end of each 3 months for the next 12 years, if money is worth 4% compounded quarterly.

32. A man borrows \$10,000. What equal payments should be made at the end of each 6 months for 10 years to discharge the loan if money is worth 6% compounded semi-annually?

33. If \$50 is deposited at the end of each 6 months in a savings account which accumulates at 2% compounded semi-annually, find the amount at the end of 20 years.

34. A loan of \$5000, with interest at 4% payable semi-annually, is to be discharged, principal and interest included, by equal payments due at the end of each 6 months for 3 years. Find the payment.

35. How much must I invest every 3 months for 10 years so that the amount at the end of that time will be \$20,000? The interest is 4% compounded quarterly.

36. In the preceding example how much must I invest at the end of each year, interest being 4% compounded yearly? What is the difference between the total payments in the two preceding examples? What is the reason for this difference?

CHAPTER 16:

INEQUALITIES

It is sometimes necessary to deal with algebraic expressions that are not equal. Thus, it may become important to decide which one of two such expressions is the greater. In this chapter, we shall discuss briefly the subject of inequalities.

212. Meaning of "greater than" and "less than".—The fundamental idea of "greater than" among numbers is that if we start with a given number b and to it add a positive number r, then the sum a is greater than b, and b is less than a. This relation between a and b is indicated by $a>b$ and also by $b<a$.

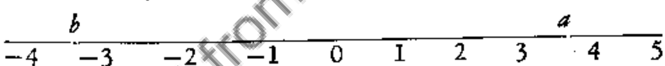
Conversely, if $a>b$, then there is a positive number r such that $a = b + r$.

$b + r > b$
If $a > b$
then $a = b + r$

The following principle is fundamental.

If in the equation $a = b + r$, r *is positive, then* $a>b$, *and if* r *is negative, then* $b>a$.

In $a>b$ we speak of a and b as the members of the inequality. If on the scale of signed numbers a is to the right of b, then $a>b$.



If x is a variable such that $a>x$, then the "range" of x consists of all numbers less than a. If $a \geqq x$, then the "range" of x consists of all numbers less than a and also including a. If $a>x>b$, then x is between a and b, while if $a \geqq x \geqq b$, x may also take the values a and b. In $a \geqq x>b$, x may take the value a but not b, while in $a>x \geqq b$, x may take the value b but not a.

In dealing with inequalities, we consider real numbers only, since complex numbers cannot be arranged in linear order, as is the case with the real numbers.

That is, complex numbers are not in ordinary linear order. Thus $2 + 3i$ is not equal to $3 + 2i$, but we cannot say that either is greater than the other.

213. Transformation of inequalities; equivalent inequalities.—If $a>b$, then the inequalities given at the right are easily proved. The principle stated on page 219 is used in each case.

If $a > b$	(1)
then:	
$a + c > b + c$	(2)
$a - c > b - c$	(3)
If m is positive:	
$ma > mb$	(4)
$\frac{a}{m} > \frac{b}{m}$	(5)
If m is negative:	
$ma < mb$	(6)
$\frac{a}{m} < \frac{b}{m}$	(7)

From $a>b$, $a = b + r$, where r is positive. Then $a + c = b + c + r$, and hence $a + c>b + c$, which is (2).

Again, $a - c = b - c + r$, and hence $a - c>b - c$, which is (3).

If in $ma = mb + mr$, m is positive, then mr is positive, and hence $ma>mb$, which is (4). Similarly, if in $\frac{a}{m} = \frac{b}{m} + \frac{r}{m}$, m is positive, then r/m is positive, and hence $a/m>b/m$, which is (5).

If in the last two above, m is negative, then mr and r/m are negative, and hence $ma<mb$ and $a/m<b/m$.

The operations on inequalities indicated above may be described in words as follows.

An inequality may be changed into an inequality subsisting in the same order by:

Adding the same number to both members.	(A)
Subtracting the same number from both members.	(S)
Multiplying both members by the same number.	(M)
Dividing both members by the same number	(D)

In (M) and (D), the multiplier and divisor must be positive. In case these are negative, the resulting inequality will subsist in the opposite order. In these steps, (A) and (S) are the inverse, each of the other, as are also (M) and (D). If we start with an identity (1) and derive an identity (2) by using a sequence of these steps, then we may start with identity (2) and obtain (1) by using the inverses of the steps used in the first process. Under these conditions the identities (1) and (2) are equivalent. If we start with a given assumed identity and from it derive an equivalent identity that is known to hold, then we know that the assumed identity also holds.

214. Identical inequalities.—Certain inequalities hold for all different values of the letters involved. These are called identical inequalities, or unconditional inequalities. It is assumed as a matter of course that only permissible substitutions are to be considered in deciding whether two inequalities are identical.

Note that it is assumed in this definition that no two letters involved may take the same value. Examples will illustrate:

Example 1. Prove that $a^2 + b^2 > 2ab$ is an unconditional inequality.

$$a^2 + b^2 = 2ab + r \quad (1)$$
$$a^2 + b^2 - 2ab = r \quad (2)$$
$$(a - b)^2 = r \quad (3)$$

Since $(a - b)^2$ is positive for all values of a and b, it follows by the steps shown at the right that r is positive and that hence $a^2 + b^2 > 2ab$ for all different values of a and b. If $a = b$ the inequality does not hold.

Example 2. Prove $a^2 + ab + b^2 > 0$.

$$a^2 + ab + b^2 = r \quad (1)$$
$$a^2 + 2ab + b^2 = r + ab \quad (2)$$
$$(a + b)^2 = r + ab \quad (3)$$

In (1) at the right, $a^2 + b^2$ is positive. Hence, if ab is positive, r is positive, and the inequality holds.

If ab is negative, add ab to both members of (1), obtaining (3) at the right. Since $r + ab = (a + b)^2$ is then positive and ab is negative, r is again positive.

EXERCISES

1. Prove $a^2 + b^2 + c^2 > ab + ac + bc$.

SOLUTION: Let $a^2 + b^2 + c^2 = ab + ac + bc + r$

Then $a^2 + b^2 + c^2 - ab - ac - bc = r$

or $\quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc = 2r$

or $\quad a^2 - 2ab + b^2 + a^2 - 2ac + c^2 + b^2 - 2bc + c^2 = 2r$

or $\quad (a - b)^2 + (a - c)^2 + (b - c)^2 = 2r$

Hence $2r$ is positive, and therefore

$a^2 + b^2 + c^2 > ab + ac + bc$.

Illustrate this inequality by substituting different values for a, b, c. Does the inequality hold when $a = b = c$?

2. Prove $a^4 + b^4 > ab(a^2 + b^2)$.

Let $a^4 + b^4 = a^3b + ab^3 + r$,

or $\quad a^4 + b^4 - a^3b - ab^3 = r$,

or $\quad a^3(a - b) + b^3(b - a) = (a^3 - b^3)(a - b) = r$,

or $\quad (a - b)^2(a^2 + ab + b^2) = r$. Use Example 2.

3. From $a > b$, $c > d$, prove $a + c > b + d$.

4. If $b > 0$, $d > 0$, and $a > b$, $c > d$, prove $ac > bd$.

5. If $a > b$, $c > d$, what follows as to the inequality of $a - c$ and $b - d$? as to the inequality of a/c and b/d?

215. *Conditional inequalities.*—We shall illustrate the treatment of conditional inequalities by means of examples.

Example 1. For what values of a does $a + 1/a > 2$ hold?

From the given inequality we obtain $(a-1)^2 = ar$ as shown at the right. Then:

$$a + 1/a = 2 + r \quad (1)$$
$$a^2 + 1 - 2a = ar \quad (2)$$
$$(a-1)^2 = ar \quad (3)$$

1. If $a = 1$, then $r = 0$.
2. If $a > 0$, $a \neq 1$, then r is positive.
3. If $0 > a$, then r is negative.
4. From the original inequality, a must be different from 0. Hence the inequality holds for all positive values of a except $a = 1$, but not for $a = 0$ or for negative values of a. Hence this inequality fails to hold for the permissible substitution $a = 1$, or for a negative, and is therefore not an identical or unconditional inequality. See the definition in §214.

Illustrate by letting a equal some positive numbers and also by letting a equal some negative numbers. Why is $a = 0$ excluded?

Example 2. Find the values of a and b for which $\dfrac{a+b}{2} > \sqrt{ab}$.

$$\frac{a+b}{2} = \sqrt{ab} + r \quad (1)$$
$$a + b = 2\sqrt{ab} + 2r \quad (2)$$
$$a - 2\sqrt{ab} + b = 2r \quad (3)$$
$$\frac{(\sqrt{a} - \sqrt{b})^2}{2} = r \quad (4)$$

To find (4) at the right, note that $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$.

From (4) we see that r is positive provided $\sqrt{a}$ and $\sqrt{b}$ are real numbers and $\sqrt{a} \neq \sqrt{b}$. Hence the inequality holds for positive values of a and b, provided $a \neq b$.

Why are negative values of a and b excluded? Suppose both a and b are negative. In this case will the given inequality contain imaginary numbers? Will the inequality $\dfrac{a+b}{2} < \sqrt{ab}$ hold in this case?

Example 3. Find the values of a and b for which $\sqrt{ab} > \dfrac{2ab}{a+b}$.

$$\sqrt{ab} = \frac{2ab}{a+b} + r \quad (1)$$
$$1 = \frac{2\sqrt{ab}}{a+b} + \frac{r}{\sqrt{ab}} \quad (2)$$
$$1 - \frac{2\sqrt{ab}}{a+b} = \frac{r}{\sqrt{ab}} \quad (3)$$
$$\frac{a+b-2\sqrt{ab}}{a+b} = \frac{r}{\sqrt{ab}} \quad (4)$$
$$\frac{(\sqrt{a} - \sqrt{b})^2}{a+b} = \frac{r}{\sqrt{ab}} \quad (5)$$

The product ab must be positive for otherwise $\sqrt{ab}$ would be imaginary. That is, a and b have the same sign. From (5) at the right $a + b$ must be positive in order that r shall be positive. Hence a and b must both be positive.

Finally, we must have $a \neq b$ to make r positive. Hence the required conditions are: $a \neq b$ and a and b both positive. Check by substituting numbers for a and b.

From pages 198, 203, 205 we have that $A = \frac{a+b}{2}$ is the arithmetic mean between a and b, $G = \sqrt{ab}$ is the geometric mean, and $H = \frac{2ab}{a+b}$ is the harmonic mean.

We have proved $\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}$ provided a and b are both positive and $a \neq b$. That is, under these conditions $A > G > H$.

Check by substituting definite numbers for a and b.

Example 4. Find conditions for which $x^3 - y^3 > (x-y)^3$.

Study the solution at the right.

$$x^3 - y^3 > (x-y)^3 \quad (1)$$
$$x^2 + xy + y^2 > (x-y)^2 \quad (2)$$
$$x^2 + xy + y^2 = x^2 - 2xy + y^2 + r \quad (3)$$
$$3xy = r \quad (4)$$

1. How is (2) obtained from (1)? What principle is used? Under what conditions is this step not justified? Is it permitted if $x < y$?
2. Under what conditions is r positive? negative? May either x or y be zero?

Check by substituting definite numbers for x and y.

Example 5. Study $(a+b)(a^3+b^3)$ and $(a^2+b^2)^2$ as to direction of inequality.

$$(a+b)(a^3+b^3) = (a^2+b^2)^2 + r \quad (1)$$
$$a^4 + ab^3 + a^3b + b^4 = a^4 + 2a^2b^2 + b^4 + r \quad (2)$$
$$ab^3 + a^3b = 2a^2b^2 + r \quad (3)$$
$$ab(b^2 + a^2) = 2a^2b^2 + r \quad (4)$$
$$b^2 + a^2 = 2ab + \frac{r}{ab} \quad (5)$$
$$(a-b)^2 = \frac{r}{ab} \quad (6)$$

Make a complete study of the work at the right.

1. In (6) may $a = 0$ or $b = 0$?
2. May a be less than b?
3. What is the sign of r if either a or b, but not both, is negative? if both are negative?

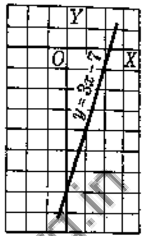
Verify your conclusions by substituting numbers for a and b.

EXERCISES

In each of the following pairs of expressions find under what conditions, if any, the first is greater than the second, and under what conditions the second is greater than the first. Under what conditions are the two expressions equal?

1. $\frac{m}{n} + \frac{n}{m}$, 2
2. $a^2 + 3b^2$, $2b(a+b)$
3. $a^3 + b^3$, $ab(a+b)$
4. $a^4 + b^4$, $ab(a^2+b^2)$
5. $\frac{a^2-b^2}{a^2+b^2}$, $\frac{a-b}{a+b}$
6. $\frac{a+b}{2}$, $\frac{2ab}{a+b}$

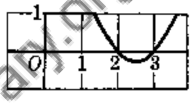
216. Graphical study of inequalities.—An expression containing a variable such as x may be positive (greater than 0) for some values of this variable and negative for other values.



Example 1. For what values of x is $3x - 7 > 0$?

Construct the graph of $y = 3x - 7$. This line crosses the x-axis for $x = 7/3$. Clearly y is positive for $x > 7/3$ and negative for $x < 7/3$. Hence $3x - 7 > 0$ for $x > 7/3$, $3x - 7 < 0$ for $x < 7/3$. This can be seen directly if we study $x - 7/3$.

Example 2. For what values of x is $x^2 - 5x + 6 > 0$?



Construct the parabola $y = x^2 - 5x + 6 = (x - 2)(x - 3)$ as shown in the figure. Then, y is negative for $2 < x < 3$, zero for $x = 2$, $x = 3$, and positive for all other values of x. That is, $x^2 - 5x + 6 > 0$ for $x < 2$ and $x > 3$.

Example 3. For what values of x is $x^2 + 5x - 10 > 0$? Consider the parabola $y = x^2 + 5x - 10$.

$$x^2 + 5x - 10 = 0$$
$$x = \frac{-5 \pm \sqrt{65}}{2}$$

From the solution at the right we know that it crosses the x-axis at $x = \dfrac{-5 - \sqrt{65}}{2}$ and $x = \dfrac{-5 + \sqrt{65}}{2}$. Between these values of x the curve lies below the x-axis. Hence $x^2 + 5x - 10 > 0$ for $x < \dfrac{-5 - \sqrt{65}}{2}$ and $x > \dfrac{-5 + \sqrt{65}}{2}$.

EXERCISES

Give values of x for which each of the following is greater than zero and also values of x, if any, for which it is less than zero.

1. $5x + 3$ **2.** $6 - 5x$ **3.** $8 + 3x$ **4.** $-9x + 5$

5. $x^2 - 8x + 12$ **6.** $x^2 + 9x - 15$ **7.** $x^2 - 3x - 9$

8. $2x^2 - 3x + 7$ **9.** $2x + 7 - x^2$ **10.** $8 - 3x^2 + 5x$

CHAPTER 17:

NUMERICAL SOLUTIONS OF EQUATIONS

The purpose of this chapter is to find real roots, either exact or approximate, of rational integral equations with numerical coefficients. The general process that is necessary is much more complicated than that used in solving equations of the first and second degree. The first part of the chapter deals with only those theoretical aspects of equations that are necessary for the approximation of real roots. These are then put together to form a general scheme for approximating numerical solutions.

217. Rational integral equations.—The general form of the rational integral equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

in which the coefficients $a_0, a_1, \ldots, a_n$ are constants and n is a positive integer. That is, all exponents are positive integers, including 0, while the coefficients may be any real or complex numbers.[1] Any of these coefficients except a_0 may be zero. The equation is then of the nth degree.

The left member of this equation will at times be denoted by $f(x)$. If b is substituted for x, the result is denoted by $f(b)$. If $f(b) = 0$, then b is a root of the equation $f(x) = 0$.

218. The fundamental theorem of algebra.—We shall assume without proof the following theorem, which is called the fundamental theorem of algebra.

A rational integral equation has at least one root.

That is, for every such equation, $f(x) = 0$, there is at least one number b such that $f(b) = 0$. This root may be real or complex. The proof of this theorem is beyond the scope of this book.

[1] In the equations for which numerical values of roots are found in this chapter, the coefficients $a_0, a_1, a_2, \ldots$ are real rational numbers, which in practice are reduced to integers.

219. ***The remainder theorem; the factor theorem.***—For convenience we shall repeat the statement and proof of these theorems. If a rational integral expression in x is to be divided by a binomial $x - a$, then clearly the process may be continued until the last remainder R does not contain x. If the quotient is Q, then we have an identity of the form shown at the right. That this is an identity means that if Q is multiplied by $x - a$, and R added to the product, the result will be exactly the expression $f(x)$. If a is substituted for x, the second member will be R. Hence the value of R may be obtained by substituting a in $f(x)$, without going through the process of dividing. This will, of course, not give the value of the quotient, Q.

$$
\begin{aligned}
f(x) &\equiv Q(x-a)+R \quad (1)\\
f(a) &\equiv Q(a-a)+R \quad (2)\\
&\equiv 0+R\\
&\equiv R
\end{aligned}
$$

Conversely, the value of $f(a)$ may be obtained without substituting by dividing $f(x)$ by $x - a$, and noting the value of R.

If a is a root of $f(x) = 0$, then $f(a) = 0$ and hence $R = 0$. That is, $x - a$ is then a factor of $f(x)$.

These results may be summarized in the following.

The remainder in $f(x) \div (x - a)$ *may be obtained by substituting* a *in* $f(x)$.

The value of $f(a)$ *may be found by taking the remainder in* $f(x) \div (x - a)$.

If a *is a root of* $f(x) = 0$, *then* $x - a$ *is a factor of* $f(x)$.

The last one of these statements is called the factor theorem.

220. ***The number of roots in an equation.***—We know, page 225, §218, that the equation $f(x) = 0$ has at least one root. Denote this root by r_1. Then $f(r_1) = 0$. But by the remainder theorem $f(x) = Q(x - r_1) + R$. Since $f(r_1) = 0$ it follows that $R = 0$. Hence $f(x)$ is exactly divisible by $x - r_1$. Let the quotient be Q_1. Then $f(x) = Q_1(x - r_1)$. But by the general theorem, page 225, §218, the equation $Q_1 = 0$ also has at least one root which we denote by r_2.

$$
\begin{aligned}
f(x) &= Q_1(x-r_1) \quad (1)\\
Q_1 &= Q_2(x-r_2) \quad (2)\\
Q_2 &= Q_3(x-r_3) \quad (3)\\
&\cdots\cdots\\
&\cdots\cdots\\
Q_{n-1} &= a_0(x-r_n) \quad (n-1)
\end{aligned}
$$

Continuing in this way, we obtain equations (1), (2), . . . $(n - 1)$ above. Hence we have,

$$f(x) = a_0\,(x - r_1)(x - r_2)(x - r_3)\ \ldots\ (x - r_n) = 0 \qquad (1)$$

Hence $r_1, r_2, \ldots, r_n$ are all roots of the equation $f(x) = 0$. That is, an equation of degree n has at least n roots. Two or more of these roots may be equal. This means that two or more of the factors $x - r$ may be repeated.

It now remains to show that equation (1) cannot have more than n roots. Suppose there is an additional root r_{n+1}. This root cannot be different from all the roots $r_1, \ldots, r_n$, for in that case, substituting r_{n+1} in (1) makes all factors different from zero and hence the product is not zero. If r_{n+1} is equal to one of the roots r_i, then we shall have an additional factor $x - r_{n+1}$ and the product will be of degree $n + 1$, while $f(x)$ is of degree n.

Hence we have the theorem:

A rational integral equation of degree n *has exactly* n *roots.*

221. Rational roots of rational integral equations.—If b/c is a root of the rational integral equation,

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

then substituting and clearing of fractions we have,

$$a_0b^n + a_1b^{n-1}c + \ldots + a_{n-1}bc^{n-1} + a_nc^n = 0$$

Since b and c have no common factor, b/c being its lowest terms, it follows from

$$a_0b^n + a_1b^{n-1}c + \ldots + a_{n-1}bc^{n-1} = - a_nc^n$$

that b is a factor of a_n.

From

$$a_1b^{n-1}c + \ldots + a_{n-1}bc^{n-1} + a_nc^n = - a_0b^n$$

it follows that c is a factor of a_0. That is, the denominator of a fractional root is a factor of the coefficient of x^n, and the numerator is a factor of the constant term.

If $a_0 = 1$, then $c = 1$ and the root must be an integer. That is, the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

has no fractional root, and any integral root is a factor of the constant term a_n.

222. ***Testing for roots of an equation.***—Let $x = b/c$ be a rational root of $f(x) = 0$.

Then by the factor theorem page 226, §219, $f(x) = \left(x - \frac{b}{c}\right)Q$ or $c\,f(x) = (cx - b)Q$. That is, $cx - b$ is a factor of $f(x)$. Hence to check whether b/c is a root of $f(x) = 0$, we divide $f(x)$ by $cx - b$. If the remainder is zero, then b/c is a root, otherwise b/c is not a root. For the purpose of facilitating the process of dividing by the binomial $cx - b$ we shall now study a short method. We shall consider the case when $c = 1$, which in practice is the more important and which makes possible a very compact method.

223. ***Synthetic division.***—Examples will illustrate this process:

Suppose we are to divide $x^5 - 7x^3 - 2x^2 + 4x - 17$ by $x - 2$.

The work arranged in the most elementary form appears as on the left below.

$$
\begin{array}{r|l}
 & x^4 + 2x^3 - 3x^2 - 8x - 12 \\
\hline
x - 2 & x^5 - 7x^3 - 2x^2 + 4x - 17 \\
 & \underline{x^5 - 2x_4} \\
 & \quad 2x^4 - 7x^3 \\
 & \quad \underline{2x^4 - 4x^3} \\
 & \qquad - 3x^3 - 2x^2 \\
 & \qquad \underline{- 3x^3 + 6x^2} \\
 & \qquad\quad - 8x^2 + 4x \\
 & \qquad\quad \underline{- 8x^2 + 16x} \\
 & \qquad\qquad - 12x - 17 \\
 & \qquad\qquad \underline{- 12x + 24} \\
 & \qquad\qquad\qquad - 41
\end{array}
\qquad
\begin{array}{r|l}
 & 1 + 2 - 3 - 8 - 12 \\
\hline
1 - 2 & 1 + 0 - 7 - 2 + 4 - 17 \\
 & \underline{1 - 2} \\
 & \quad 2 - 7 \\
 & \quad \underline{2 - 4} \\
 & \qquad - 3 - 2 \\
 & \qquad \underline{- 3 + 6} \\
 & \qquad\quad - 8 + 4 \\
 & \qquad\quad \underline{- 8 + 16} \\
 & \qquad\qquad - 12 - 17 \\
 & \qquad\qquad \underline{- 12 + 24} \\
 & \qquad\qquad\qquad - 41
\end{array}
$$

It is clear that the x's may be omitted in the written work since the coefficients determine completely both the quotient and the remainder. The term in x^4, which is lacking in the dividend, must be indicated by writing the coefficient 0, as on the right above.

Still further simplification is possible since the highest power in the dividend must disappear at the first step, and the highest power in each remainder must disappear as the corresponding step in the division is taken. Hence these may be omitted in the products.

The work may be made more compact if the coefficients in the dividend are not brought down, and the numerals -4, $+6$, $+16$, and $+24$, are written directly under them. Finally we may change the sign of the -2 in the divisor, and then add each product instead of subtracting it. Then we have:

$$\begin{array}{r|rrrrrr} +2) & 1 & 0 & -7 & -2 & +4 & -17 \\ & & 2 & +4 & -6 & -16 & -24 \\ \hline & & 2 & -3 & -8 & -12 & -41 \end{array}$$

From this computation it is known that:

(a) the quotient is of the fourth degree and the coefficient of x^4 is 1;

(b) the coefficients in order of the remaining terms of the quotient are the numbers below the lines (except the last);

(c) the last number below the lines, -41, is the remainder.

Hence the quotient is $x^4 + 2x^3 - 3x^2 - 8x - 12$ and the remainder is -41. By comparing the three forms of the division shown above it is easily seen why they all lead to the same result.

The last of these forms is always available when the divisor is of the form $x - a$, a being either positive or negative.

Example. Divide $x^7 - 3x^4 + 2x^2 - 9x + 8$ by $x + 3$.

$$\begin{array}{r|rrrrrrrr} -3) & 1 & 0 & 0 & -3 & 0 & +2 & -9 & +8 \\ & & -3 & +9 & -27 & +90 & -270 & +804 & -2385 \\ \hline & & -3 & +9 & -30 & +90 & -268 & +795 & -2377 \end{array}$$

Hence the quotient is $x^6 - 3x^5 + 9x^4 - 30x^3 + 90x^2 - 268x + 795$ and the remainder is -2377.

EXERCISES

Find the quotient and remainder in each of the following:

1. $(x^3 - 4x^2 + 2x - 7) \div (x - 2)$
2. $(x^4 + 2x^3 - 7x + 5) \div (x + 3)$
3. $(x^4 + 5x^2 - 3x + 3) \div (x - 4)$
4. $(x^4 - 9x + 12) \div (x + 4)$
5. $(x^5 + 3x^4 - 2x + 8) \div (x - 5)$
6. $(x^5 - x^4 + x^3 - 9) \div (x - 3)$
7. $(x^5 - 7x^3 + 8x^2 - 11) \div (x + 6)$
8. $(x^6 - 9x^4 + 2x^2 - 6) \div (x - 7)$
9. $(x^5 + 2x^3 + 4x^2 - 7x - 8) \div (x - 6)$
10. $(x^7 - 3x^5 + 4x^3 + 2x - 6) \div (x + 1)$
11. $(x^6 + 7x^4 - 3x^3 - x + 6) \div (x - 4)$
12. $(x^5 - 4x^4 + 2x^3 - 12) \div (x + 2)$
13. $(x^7 + 2x^6 - 3x^5 + 4) \div (x - 1)$
14. $(x^3 + 5x^2 + 7x - 8) \div (x + 2)$
15. $(x^4 - 3x^3 + 14) \div (x - 3)$
16. $(x^3 + 6x - 9) \div (x + 3)$

224. Finding rational roots.—We have seen, page 227, §221, that if $x^n + a_1x^{n-1} + \cdots + a_n = 0$ has an integral root, then this root is a divisor of a_n. Suppose we wish to test the equation at the right for integral roots. We know at once that the only possible roots are ± 1, ± 3, since these are the only divisors of -3. We can see at a glance that 1 or -1 is not a root since substituting either of these obviously does not reduce the left member to zero. We then need to test for 3 and -3. Dividing by $x - 3$ gives 0 as a remainder and hence $x = 3$ is a root.

$$x^3 - 2x^2 - 2x - 3 = 0$$

The quotient is $x^2 + x + 1$, and clearly no integer will reduce this to zero. Hence 3 is the only integral root of this equation.

3 | 1	-2	-2	-3
	$+3$	$+3$	$+3$
1	1	1	0

We know further from §221 that the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

has no rational fractional root if $a_0 = 1$. In case $a_0 \neq 1$, there may be a fractional root b/c in which c is a factor of a_0 and b a factor of a_n. Consider equation (1) at the right. We know that $\pm 3/2$ are the only possible fractional rational roots. By dividing we find that $2x - 3$ is a factor and that hence $x = 3/2$ is a root. Clearly $x = -3/2$ is not a root.

$$2x^3 - 9x^2 + 10x - 3 = 0 \quad (1)$$
$$(2x - 3)(x^2 - 3x + 1) = 0$$
$$x = 3/2$$

Equations of this type occur less frequently and our more important case is finding integral roots when $a_0 = 1$.

EXERCISES

Search for integral roots in the following equations.

1. $x^3 - x^2 - x - 2 = 0$
2. $x^4 + x^3 - x^2 + 1 = 0$
3. $x^3 - 7x - 6 = 0$
4. $x^3 + 2x^2 + 3x - 6 = 0$
5. $x^3 - 4x^2 + 2x - 2 = 0$
6. $x^4 + 3x^3 + 2x^2 - 2x - 4 = 0$
7. $x^3 - 11x^2 + 31x - 6 = 0$
8. $x^5 + 2x^4 - x^2 + 4 = 0$
9. $x^4 - x^3 - 2x^2 + x - 2 = 0$
10. $x^3 + 7x^2 - 2x + 2 = 0$
11. $x^3 + 4x^2 - x - 4 = 0$
12. $x^3 + 2x^2 - 2x + 3 = 0$
13. $x^4 + 5x^3 - 7x^2 + 33x + 10 = 0$
14. $x^5 + 5x^4 - 5x^2 - 24x + 5 = 0$
15. $x^4 - 6x^3 - 3x^2 + 22x - 24 = 0$
16. $x^4 + 3x^3 - 8x^2 - 12x + 16 = 0$
17. $x^5 + 3x^4 - 45x^3 + 82x^2 + 6x - 27 = 0$
18. $x^6 - 5x^5 + 5x^2 - 4 = 0$

225. *Locating irrational real roots.*—The general "location" of the roots of an equation with numerical coefficients may be found by constructing a graph. An example will illustrate:

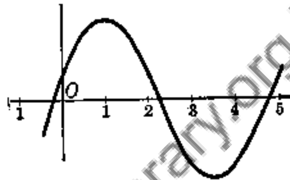
Example 1. Locate the roots of the equation $x^3 - 7x^2 + 10x + 2 = 0$.

SOLUTION: We shall first construct a graph of the equation

$$y = x^3 - 7x^2 + 10x + 2.$$

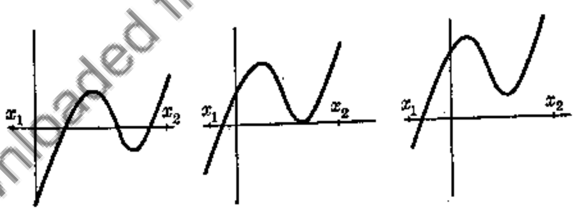
This equation is satisfied by the following pairs of numbers.

x	0	1	2	3	4	5	6	−1
y	2	6	2	−4	−6	2	26	−16



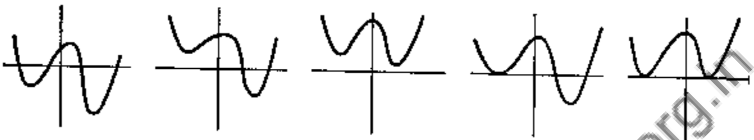
By plotting these points and drawing a smooth curve through them, we have the figure at the right. It is now clear that there is a value of x between −1 and 0 for which $y = 0$. There is also one such value of x between 2 and 3 and one between 4 and 5. This gives a rough "location" of the three real roots of this equation.

We can see that, in general, if $f(x_1)$ is negative and $f(x_2)$ is positive, then $f(x) = 0$ must have at least one real root between x_1 and x_2. In fact, it must have an odd number of roots between these two values of x. The following sketches show how the graphs may behave.



In the first figure there are three distinct roots between x_1 and x_2. In the next figure there are likewise three roots between x_1 and x_2, but two of them are equal. That is, the curve just touches the x-axis at one point, but does not cross it. This represents a double root (two equal roots). In the last figure there is only one root between x_1 and x_2. These graphs represent a cubic.

The following graphs represent fourth degree equations. In this case the graph goes up indefinitely on both sides. It may cross the axis in four points or in two points, or in no point. Or it may cross the axis in two points and "touch" it at one point, or it may even touch it in two points.



The following is a general theorem which is quite obvious, but a rigorous proof of it is beyond the scope of this book.

If the value of a rational integral function changes sign as x changes from x_1 to x_2, there is an odd number of values of x between x_1 and x_2 for which the function is equal to zero.

It is, namely, obvious that to pass from one side of a line to the other we must cross it an odd number of times.

We shall now study two transformations of an equation that are necessary in approximating its real roots.

226. ***Increasing or decreasing the roots of an equation by a given number.***—One of the important steps in approximating the roots of an equation consists in decreasing its roots by a given number.

Consider the equation

$$x^3 - 9x^2 + 21x - 5 = 0 \tag{1}$$

Substitute $x + 1$ for x in this equation, obtaining

$$(x + 1)^3 - 9(x + 1)^2 + 21(x + 1) - 5 = 0 \tag{2}$$

Let r be a root of (1). Then $r - 1$ is a root of (2).

For
$$r^3 - 9r^2 + 21r - 5 = 0 \tag{3}$$
and
$$(r - 1 + 1)^3 - 9(r - 1 + 1)^2 + 21(r - 1 + 1) - 5 = 0 \tag{4}$$
since (4) is identical with (3).

Hence each root of (2) is 1 less than the corresponding root of (1).

Expanding the terms of (2) and collecting terms, we have

$$x^3 - 6x^2 + 6x + 8 = 0 \tag{5}$$

When (2) is of higher degree, this reduction may be complicated, and it therefore becomes important to find a simple way of making it. Equation (2) reduces to the form

$$x^3 + Bx^2 + Cx + D = 0 \qquad (6)$$

and we need to find B, C, and D (which we found to be -6, 6, and 8 respectively).

If in (6) x is replaced by $x - 1$, we shall increase its roots by 1 and we have

$$(x - 1)^3 + B(x - 1)^2 + C(x - 1) + D = 0 \qquad (7)$$

whose roots are the same as those of (1). That is,

$$(x - 1)^3 + B(x - 1)^2 + C(x - 1) + D \text{ and } x^3 - 9x^2 + 21x - 5$$

are identical.

But dividing $(x - 1)^3 + B(x - 1)^2 + C(x - 1) + D$ by $x - 1$ gives a remainder D, and dividing $x^3 - 9x^2 + 21x - 5$ by $x - 1$ must give the same remainder. That is, D can be found by dividing (1) by $x - 1$.

In this division the quotient is $(x - 1)^2 + B(x - 1) + C$. Dividing this quotient by $x - 1$, we find that the next quotient is $x - 1 + B$ and the remainder is C. Dividing again by $x - 1$ gives the remainder B.

In this way the following theorem may be proved.

If the left member of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \qquad (1)$$

is divided successively by $x - a$, *the remainders* $b_n, b_{n-1}, \dots b_{n-n+1}$, b_0 *will be the coefficients of the equation*

$$b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \dots + b_{n-1}x + b_n = 0 \qquad (2)$$

each of whose roots is less by a *than the corresponding root of (1).*

EXERCISES

1. Divide $x^3 - 9x^2 + 21x - 5$ by $x - 1$, then divide the quotient by $x - 1$, and then this quotient by $x - 1$. How do the remainders compare with the coefficients in (5) above?

2. If it is given that 5 is a root of (1), page 232, find the remaining roots of this equation.

3. If these roots of (1) are all decreased by 1, they should be the roots of (5) above. Check to see whether this is the case.

227. Practice in locating real roots of an equation.—In the actual work of locating roots we use the remainder theorem and carry out the division by the synthetic method.

Example. Find consecutive integers between which lie roots of

$$x^5 - 5x^4 - 10x^3 + 50x^2 + 8x - 52 = 0.$$

SOLUTION: Designate the polynomial by $f(x)$.

Evaluate $f(x)$ for $x = 1$, $x = 2$ . . . as follows.

$$\begin{array}{lrrrrrr} f(1): & 1 & -5 & -10 & +50 & +8 & -52 \\ & & +1 & -4 & -14 & 36 & 44 \\ \hline & & -4 & -14 & 36 & 44 & -8 \end{array}$$

$$\begin{array}{lrrrrrr} f(2): & 1 & -5 & -10 & +50 & +8 & -52 \\ & & 2 & -6 & -32 & 36 & 88 \\ \hline & & -3 & -16 & 18 & 44 & 36 \end{array}$$

$$\begin{array}{lrrrrrr} f(3): & 1 & -5 & -10 & +50 & +8 & -52 \\ & & 3 & -6 & -48 & 6 & 42 \\ \hline & & -2 & -16 & 2 & 14 & -10 \end{array}$$

Continuing in this manner we have:

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	-4	100	-6	-52	-8	36	-10	-116	-12	932

Hence we know there is at least one root between $x = -3$ and $x = -2$. There is also at least one root between $x = -2$ and $x = -1$, between $x = 1$ and $x = 2$, between $x = 2$ and $x = 3$, and between $x = 5$ and $x = 6$.

Since this equation has only five roots we know there is exactly one root on each of these intervals and that there is no root greater than $x = 6$ or less than $x = -3$.

EXERCISES

For each of the following equations find consecutive integers between which the equation has a root.

1. $x^3 - 4x^2 + 6x - 3 = 0$
2. $x^3 + 6x^2 - 2x + 1 = 0$
3. $x^3 - 19x + 7 = 0$
4. $x^3 + 8x^2 - 10 = 0$
5. $x^4 + 2x^3 - 6x^2 - 3x + 5 = 0$
6. $x^4 - 8x^2 + 2x - 9 = 0$
7. $x^4 + 7x^3 - 19x + 2 = 0$
8. $x^4 - 3x^3 + 6x^2 - 8 = 0$
9. $x^5 + 6x^3 - 10x^2 + 3 = 0$
10. $x^5 - 2x^4 + 8x^2 - 6x - 7 = 0$

228. *Changing the signs of the roots of an equation.*—If r is a root of the equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \tag{1}$$

then $-r$ is a root of

$$a_0x^3 - a_1x^2 + a_2x - a_3 = 0. \tag{2}$$

For, substituting r in (1) we have

$$a_0r^3 + a_1r^2 + a_2r + a_3 = 0 \tag{3}$$

Substituting $-r$ in the left member of (2) gives

$$a_0(-r)^3 - a_1(-r)^2 + a_2(-r) - a_3 = -a_0r^3 - a_1r^2 - a_2r - a_3,$$

which by (3) is equal to zero.

In this manner it may be proved that for any positive integral value of n the roots of the equation

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots (-1)^n a_n = 0$$

are the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

with their signs changed.

Hence we have the rule:

To change the signs of the roots of an equation change the signs of alternate terms beginning with the second term from the left.

Notice that if one or more of the terms in the equation are missing, these must be counted in changing the signs.

Thus to change the signs of the roots in

$$a_0x^5 + a_2x^3 + a_4x + a_5 = 0,$$

we may write

$$a_0x^5 + 0\cdot x^4 + a_2x^3 + 0\cdot x^2 + a_4x + a_5 = 0.$$

EXERCISES

1. What are the roots of $(x - 1)(x + 2)(x - 3)(x - 4) = 0$? Multiply the left member to form a polynomial, put it equal to zero, and change the equation so that its roots shall be those of the given equation with their signs changed. Check this equation for the required roots.

Transform the following so as to change the signs of their roots.

2. $x^3 - x + 7 = 0$
3. $x^4 - 3x^3 + x - 3 = 0$
4. $x^5 + 2x^3 + 5x + 9 = 0$
5. $x^7 - 2x^6 - 3x^4 + 5x - 8 = 0$

229. General scheme for approximating a root of an equation.—Consider the equation $x^3 - 7x^2 + 10x + 2 = 0$. On page 231 we found that this equation has a root between 2 and 3. We then proceed as follows.

$$x^3 - 7x^2 + 10x + 2 = 0 \quad (1)$$
$$x^3 + B_1x^2 + C_1x + D_1 = 0 \quad (2)$$
$$x^3 + B_2x^2 + C_2x + D_2 = 0 \quad (3)$$
$$x^3 + B_3x^2 + C_3x + D_3 = 0 \quad (4)$$

STEP 1. As in §226 decrease the roots of this equation by 2, obtaining an equation $x^3 + B_1x^2 + C_1x + D_1 = 0$ (2), which has a root between 0 and 1.

STEP 2. As on page 231 locate a root of (2). Suppose this equation has a root between .3 and .4. Then (1) has a root between 2.3 and 2.4.

STEP 3. Decrease the roots of (2) by .3, obtaining equation (3).

STEP 4. Locate a root of (3). Suppose we find a root between .02 and .03. Then (2) has a root between .32 and .33 and (1) has a root between 2.32 and 2.33. Continuing in this way, we may approximate the root as closely as we please.

In the above we find that after decreasing the roots in (1) by 2 the result is $x^3 - x^2 - 6x + 2 = 0$, which has a root between 0 and 1. We then locate this root more definitely as between .3 and .4. Decreasing the roots of the equation by .3 gives $x^3 - .1x^2 - 6.33x + 0.137 = 0$, which has a root between .02 and .03.

If a root is found to be negative, transform the equation so as to change the signs of its roots and then approximate the corresponding positive root.

You should take pains to make clear to yourself the nature of these steps and the reason for taking them. If you do, you will understand the concise method for solving such equations that we shall study on the next page. This method may be used, for instance, to approximate any real root of a number. Thus we may approximate $\sqrt[7]{217}$ by approximating a root of $x^7 - 217 = 0$.

230. Horner's method.—The method outlined above for approximating real roots of an equation is called Horner's method. We shall start by solving two equations whose solution we can find by methods that we have already studied.

Example 1. Approximate a positive root of $x^2 - 2x - 2$.

STEP 1. By the method of §227 we find that this equation has a root between 2 and 3.

STEP 2. Transform the equation so as to decrease its roots by 2, obtaining $x^2 + 2x - 2 = 0$, and so on.

The transformed equation,	Divisors				Figures in root
	2	1	− 2	− 2	(2.
			2	0	
			0	− 2	
			2		
			2		
$x^2 + 2x - 2 = 0$	.7	1	2	− 2	(.7
			.7	1.89	
			2.7	− .11	
			.7		
			3.4		
$x^2 + 3.4x - .11 = 0$	.03	1	3.4	− .11	$\frac{.11}{3.4} = .03$
			.03	.1029	
			3.43	− .0071	
			.03		
			3.46		
$x^2 + 3.46x - .0071 = 0$	.002	1	3.46	− .0071	$\frac{.0071}{3.46} = .002$
					root is 2.732

Example 2. Approximate a root of $x^3 - 3 = 0$.

Divisors					Figures in root
1	1	0	0	−3	(1.
		1	1	1	
		1	1	−2	
		1	2		
		2	3		
		1			
		3			(.4
.4	1	3	3	−2	
		.4	1.36	1.744	
		3.4	4.36	− .256	
		.4	1.52		
		3.8	5.88		
		.4			
		4.2			$\frac{.256}{5.88} = .04$ (.04

Continue to three decimal places.

Example 3. Approximate a positive root of $x^3 - 6x^2 + x + 9 = 0$.

STEP 1. We find that there is a root between $x = 1$ and $x = 2$ and one between $x = 5$ and $x = 6$.

We shall approximate the root between $x = 1$ and $x = 2$.

STEP 2. Transform the equation so as to decrease its roots by 1, obtaining $x^3 - 2x^2 - 8x + 5 = 0$.

This equation has a root between .5 and .6.

The work may be arranged as follows.

Divisors					Figures in root
1	1	−6	+1	+9	(1.
		1	−5	−4	
		−5	−4	+5	
		1	−4		
		−4	−8		
		1			
		−3			
.5	1	−3	−8	+5	(.5
		.5	−1.25	−4.625	
		−2.5	−9.25	.375	
		.5	−1.00		
		−2.0	−10.25		
		.5			
		−1.5			
.03	1	−1.5	−10.25	.375	(.03

Describe the steps in the above.

Example 4. Approximate the fifth root of 89.

SOLUTION: This is equivalent to solving $x^5 - 89 = 0$.

						Figures in root
1	0	0	0	0	−89	(2.
	2	4	8	16	32	
	2	4	8	16	−57	
	2	8	24	64		
	4	12	32	80		
	2	12	48			
	6	24	80			
	2	16				
	8	40				
	2					
	10					
1	+10	+40	+80	+80	−57	(.4

Continue this process to find root correct to three places of decimals.

Note the method suggested on page 237 for finding the successive figures in the roots. This method is unreliable in the first steps of the process. Thus in Example 4 on page 238 this would give $57/80 = .7$ (approximately), while the true figure is .4. A little study will enable you to find the figures correctly without too much trouble. For the later steps the method suggested is quite accurate. Note that the last figures in the successive reduced equations all have the same sign. Thus in Example 1 these figures are $-2, -2, -.11, -.0071$. In Example 2 they are $-3, -2, -.256$.

Example 5. Approximate to four places of decimals the negative root of

$$x^3 - 6x^2 + x + 9 = 0 \tag{1}$$

SOLUTION: By the method of §228 transform this equation so as to change the signs of its roots, obtaining

$$x^3 + 6x^2 + x - 9 = 0 \tag{2}$$

The purpose in making this change is to avoid little differences in the work when a negative root is approximated.

The work of approximating the root of this equation which lies between $x = 1$ and $x = 2$ is then as follows.

Divisors					Figures in root
1	1	6	1	−9	(1.
		1	7	8	
		7	8	−1	
		1	8		
		8	16		
		1			
		9			
.06	1	9	16	−1	(.06
		.06	.5436	.992616	
		9.06	16.5436	− .007384	
		.06	.5472		
		9.12	17.0908		
		.06			
		9.18			
.0004	1	9.18	17.0908	− .007384	(.0004
		.0004	.00367216	.006837788864	
		9.1804	17.09447216	− .000546211136	
		.0004	.00367232		
		9.1808	17.09814448		
		.0004			
		9.1812			

Hence -1.0604 is a four-place approximation of a root of (1).

Example 6. Approximate a root of $x^4 - 6x^2 + 3x - 7 = 0$

SOLUTION: We find that there is a root between 2 and 3.

The work is then as follows.

Divisors						Figures in root
2	1	0	−6	+3	−7	(2
		2	4	−4	−2	
		2	−2	−1	−9	
		2	8	12		
		4	6	11		
		2	12			
		6	18			
		2				
		8				
.4	1	8	18	11	−9	(.4
		.4	3.36	8.544	7.8576	
		8.4	21.36	19.544	−1.1424	
		.4	3.52	9.952		
		8.8	24.88	29.496		
		.4	3.68			
		9.2	28.56			
		.4				
		9.6				
	1	9.6	28.56	29.496	−1.1424	

Continue to three decimal places.

EXERCISES AND PROBLEMS

Find the specified roots accurate to three decimal places.

1. The root of $x^3 - x^2 - 2x + 1 = 0$ between 1 and 2.
2. The root of $x^4 - 5x^3 + 10x^2 - 7x = 5$ between 0 and -1.
3. The root of $x^3 - 7x^2 + 15x + 89 = 0$ between -2 and -3.
4. The root of $x^3 - 7x + 7 = 0$ between -3 and -4.
5. The root of $x^4 - x^3 - 1 = 0$ between 1 and 2.
6. The larger positive root of $x^3 - 4x^2 + 3x + 1 = 0$.
7. The positive root of $x^3 - x^2 - x - 4 = 0$.
8. The negative root of $x^4 - x^2 - 3 = 0$.
9. The root of $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ between 2 and 3.
10. By using Horner's method, find $\sqrt[3]{16}$ correct to three decimal places.
11. By using Horner's method, find $\sqrt[4]{27}$ correct to three decimal places.
12. The root of $x^3 + 10x^2 + 6x - 120 = 0$ between 2 and 3.
13. The root of $x^2 - 2x - 5 = 0$ between 3 and 4.
14. The root of $x^4 - 2x^3 + 21x - 23 = 0$ between 1 and 2.

15. The root of $2x^3 + 3x^2 - 4x - 10 = 0$ between 1 and 2.
16. The root of $x^3 - 46x^2 - 36x + 18 = 0$ between 0 and 1.
17. The root of $3x^3 + 5x - 40 = 0$ between 2 and 3.
18. The root of $x^3 + 10x^2 + 8x - 120 = 0$ between 2 and 3.
19. By using Horner's method, find $\sqrt[4]{13}$ correct to four decimal places.
20. By using Horner's method, find $\sqrt[3]{5}$ correct to three decimal places.
21. Find the real roots of $x^3 - 3x - 1 = 0$.
22. Find the real roots of $x^3 - 22x - 24 = 0$.
23. The equation $x^3 - 4x^2 - 2x + 8 = 0$ has a root between 1 and 2. Approximate this root to four places of decimals.
24. Locate a positive root of $x^3 + 2x^2 - x - 10 = 0$ and approximate it to four places of decimals.
25. Approximate $\sqrt[3]{7}$ to four places of decimals.
26. Find consecutive integers between which are roots of $x^4 - 7x^3 + 8x^2 + 28x - 50 = 0$. Approximate the largest of these roots to three places of decimals.
27. A box 8 inches wide, 6 inches deep, and 16 inches long is to be doubled in capacity by increasing its width and depth each by x and the length by $2x$. Find the value of x to two places of decimals.

28. A horizontal beam of length l is held fast at one end and rests loose at the other end. A weight resting on this beam will cause the maximum bending if it rests at a distance $(k - 1)l$ from the loose end provided k has the smallest positive value that satisfies the equation $2k^3 - 3k + 1 = 0$. Find this value of k to two places of decimals.
29. A not uncommon type of problem in finance leads to equations like the following.

$$3000(1 + i)^4 + 18000(1 + i)^2 + 6000(1 + i) = 30000.$$

It is required to find the value of i (rate of interest) to two places of decimals. To do this find a value of $1 + i$ correct to three places of decimals. Notice that negative values of $1 + i$ have no meaning in this problem.

30. The dimensions in feet of a rectangular box are in an arithmetic sequence whose common difference is 1 foot. If its volume is 50 cubic feet, find the dimensions of the box, correct to two decimal places.
31. A rectangular box is made from a piece of sheet metal 6 inches by 10 inches by cutting equal squares from the corners and turning up the sides. Find the side of the square to be cut out, correct to two decimal places, if the volume of the box is to be 20 cubic inches.
32. The edges of a rectangular box are 3 feet, 4 feet, and 9 feet long. To double the volume, the first two dimensions will be increased and the third will be decreased by the same amount. Find the new dimensions.

231. Approximating solutions by the method of false position.—The following Examples illustrate a method of very general usefulness in finding approximate solutions of problems.

Example 1. Approximate the root of $x^3 - 7x^2 + 4x - 6 = 0$ which lies between 6 and 7.

SOLUTION: We find by synthetic division that $f(6) = -18$ and $f(7) = 22$. Hence we assume that the root is near 6.5. Suppose $x = 6.5 + h$ is the exact root. Then $(6.5 + h)^3 - 7\,(6.5 + h)^2 + 4\,(6.5 + h) - 6 = 0$. Since h is a small fraction, h^2 and h^3 are so small that we may omit all terms in which they are factors. Expanding and omitting these terms we have

$$(6.5)^3 + 3(6.5)^2h - 7(6.5)^2 - 7(13)h + 4(6.5) + 4h - 6 = 0.$$

Solving this equation for h gives, $h = .0283$ which is an approximate value of h. Then 6.5283 is a close approximation of the root. Supposing that $6.5283 + h_1$ is the exact root and proceeding as above we have, $h_1 = -.00025$. Hence $6.5283 - .00025 = 6.52805$ is the next approximation which is certainly correct to three decimals and very likely to five.

Example 2. Find the approximate value of the coordinates of the intersection point of the curves $y^2 = x^3$ and $(x - 2)^2 + (y + 1)^2 = 25$ which lies in the first quadrant.

SOLUTION: Constructing the graphs of these equations it is evident that the coordinates of the required intersection point are nearly $x = 2.5$, $y = 4$.

Let $2.5 + h$ and $4 + k$ be the exact coordinates.

Substituting these values for x and y in the equations we have

$$(4 + k)^2 = (2.5 + h)^3$$
$$(.5 + h)^2 + (5 + k)^2 = 25.$$

Expanding and omitting second and third powers of h and k we have,

$$8k - 18.75h = -.375$$
$$h + 10k = -.25$$

which gives, $h = .008951$, $k = -.0259$.

Hence, $x = 2.508951$, $y = 3.9741$ is our next approximation.

Substituting $x = 2.5089 + h_1$, $y = 3.9741 + k_1$, and solving as before we find $h_1 = .00002$, $k_1 = -.00007$.

Hence $x = 2.50892$, $y = 3.97403$.

By this method a pair of equations of higher degree may readily be solved, a task which by other methods would be very difficult.

CHAPTER 18:

THEORY OF EQUATIONS

Several of the theorems that usually come under the heading of this chapter were studied in Chapter 17. In the present chapter we shall study some additional properties of equations of which use is made in later study of mathematics.

232. ***Conjugate roots.***—We shall first prove the following.

If a rational integral equation with real coefficients has a complex root a + bi, *then* a − bi *is also a root of this equation.*

PROOF. If $a + bi$ is a root of $f(x) = 0$, then $x - a - bi$ is a factor of $f(x)$. If $a - bi$ is also a root, then $x - a + bi$ is a factor of $f(x)$. In any case, $(x - a - bi)(x - a + bi) = (x - a)^2 + b^2 = x^2 - 2ax + a^2 + b^2$.

Dividing $f(x)$ by this second degree expression gives (1) at the right, in which R_1 and R_2 are real numbers.

$$f(x) = Q(x^2 - 2ax + a^2 + b^2) + R_1x + R_2 \quad (1)$$
$$R_1(a + bi) + R_2 = 0 \quad (2)$$
$$R_1a + R_2 = 0 \quad (3)$$
$$R_1bi = 0 \quad (4)$$

By hypothesis $a + bi$ is a root. Hence, substituting this value in (1) gives $R_1(a + bi) + R_2 = 0$. Note that $x - a - bi$ is one of the factors of $x^2 - 2ax + a^2 + b^2$. In equation (2) the real part of the remainder, $R_1a + R_2$, must be zero, as must also the imaginary part R_1bi. But b cannot be zero, since the number $a + bi$ is assumed not to be real. Hence $R_1 = 0$, and therefore $R_2 = 0$. If $a - bi$ is substituted in (1), R_1 and R_2 remain the same and hence $R_1(a - bi) + R_2 = 0$. Therefore $a - bi$ is also a root.

We speak of the roots $a + bi$ and $a - bi$ as conjugate complex roots. By using the theorem that if $a_1 + \sqrt{b_1} = a_2 + \sqrt{b_2}$, a_1 and a_2 being rational and $\sqrt{b_1}$ and $\sqrt{b_2}$ being surds, then $a_1 = a_2$ and $b_1 = b_2$, we can prove exactly as above that if $a + \sqrt{b}$ is a root then $a - \sqrt{b}$ is also a root.[1] These are conjugate real roots.

[1] This theorem may also be proved in the same manner that the theorem for complex roots is proved in §233.

233. ***Alternate proof of the theorem on complex conjugate roots.***—Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ be the equation. By hypothesis it is satisfied by $x = a + bi$. On substituting this value of x the equation reduces to the form $P + Qi = 0$ in which P represents the algebraic sum of all terms that do not contain the factor i, and Q represents the algebraic sum of the cofactors of i in all terms that contain this factor.

In this substitution the i disappears in all terms in which it is raised to an even power and remains in all terms in which it is raised to an odd power.

Moreover, $(i)^k = (-i)^k$ for all even values of k and $(i)^k = -(-i)^k$ for all odd values of k.

Hence if $a - bi$ instead of $a + bi$ is substituted, all terms in which i disappears will be unchanged while all terms in which it remains will be changed in sign. Hence the result will be $P - Qi$, P and Q being the same as above.

But $P + Qi = 0$ and hence $P = 0$ and $Q = 0$. Therefore, $P - Qi = 0$ and the equation is satisfied when $a - bi$ is substituted.

EXERCISES

1. Prove that an expression

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

in which $a_0, a_1, \ldots a_n$ are real numbers and n is even, has $\frac{n}{2}$ factors of the form

$$ax^2 + bx + c, \text{ in which } a, b, c \text{ are real numbers.}$$

2. Prove that in case n is an odd number the expression in exercise 1 has one factor of the form $ex + d$ and $\frac{n-1}{2}$ factors of the form $ax^2 + bx + c$ in which a, b, c, d, e are real numbers.

3. If it is known that $2 + 3i$ is a root of the equation $x^3 - 6x^2 + 21x - 26 = 0$, find the other roots of this equation.

4. Find all the roots of the equation $x^4 + x^3 + 4x^2 - 3x + 5 = 0$, if it is given that $-1 + 2i$ is one of its roots.

5. Find all the roots of $x^5 - x^3 + 4x^2 - 2x + 4 = 0$ if it is given that $1 + i$ and i are roots of this equation.

6. Can you prove the conjugate root theorem for real roots by using the process of §233?

234. *Limit of real roots of an equation.*—Consider the equation

$$x^4 - 2x^3 + 5x^2 - 7x + 4 = 0.$$

Dividing the left member successively by $x - 1$, $x - 2$, . . . we have

$$\begin{array}{rrrrrr} 1) & 1 & -2 & +5 & -7 & +4 \\ & & 1 & -1 & 4 & -3 \\ \hline & & -1 & 4 & -3 & 1 \end{array} \qquad (1)$$

$$\begin{array}{rrrrrr} 2) & 1 & -2 & +5 & -7 & +4 \\ & & 2 & 0 & 10 & 6 \\ \hline & & 0 & 5 & 3 & 10 \end{array} \qquad (2)$$

From this we have

$$x^4 - 2x^3 + 5x^2 - 7x + 4 = (x - 1)(x^3 - x^2 + 4x - 3) + 1 \qquad (1')$$

and

$$x^4 - 2x^3 + 5x^2 - 7x + 4 = (x - 2)(x^3 + 5x + 3) + 10 \qquad (2')$$

It is evident that there is no value of x greater than 2 which makes the right member of (2′) equal to zero. Hence the given equation can have no root greater than 2.

The example above is a special case of the following.

Theorem: If in the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

a_0 *is positive, and if on dividing the left member by* $x - a$, *where* a *is positive, all the coefficients in the quotient and also the remainder are positive, then the equation has no root greater than* a.

The proof is obvious and is left for the student.

To determine the lower limit of negative roots change the signs of the roots by the method of §228 and proceed as above.

EXERCISES

1. Find the smallest integer b such that the equation $x^3 - 5x^2 + 3x - 7 = 0$ has no real root greater than b. Note that this equation has a real root between b and $b - 1$.

2. Find the two consecutive integers between which lies the greatest real root of $x^4 + 2x^3 - 10x - 12 = 0$.

3. Find the two consecutive negative integers between which lies the root of $x^4 + 6x^3 - 3x^2 + 15 = 0$ with the greatest numerical value.

235. ***Descartes' rule of signs.***—If in the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

all the coefficients are positive, then no positive value of x can make the left member zero and hence the equation can have no positive root.

Consider this equation when a_0 is positive, while the other coefficients may be positive or negative. In passing along the series of coefficients

$$a_0, a_1, a_2, \ldots a_{n-1}, a_n,$$

from left to right for instance, there will be changes from positive to negative coefficients and again from negative to positive. Each such change is called a variation in signs of the coefficients.

Thus in the equation

$$x^4 + 3x^3 - 2x^2 - 6x - 2 = 0$$

there is one variation in signs, while in

$$x^4 - 3x^3 - 2x^2 + 6x + 2 = 0 \text{ and } x^4 - 3x^3 + 2x^2 - 6x + 2 = 0$$

there are 2 and 4 variations respectively.

Descartes' rule of signs is stated in the following.

Theorem: If there are k *variations of signs in an equation, the equation cannot have more than* k *positive roots.*

PROOF: Let $r_1, r_2, \ldots r_m$ be the positive real roots of the equation

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0.$$

Then $x - r_1, x - r_2, \ldots$ are factors of the left member and this member may be written

$$(x - r_1)(x - r_2) \ldots (x - r_m)Q(x)$$

where $Q(x)$ is a polynomial in x.

Let the variations in signs in $Q(x)$ be represented schematically by

$$+ \ldots + - \ldots - + \ldots + - \ldots - + \ldots +$$

where the dots indicate that there is no variation of signs between the two signs on either side of them. Suppose $Q(x)$ is multiplied by $x - r_1$, where r_1 is positive. The signs in this operation are indicated as shown on the next page.

If all the ambiguous signs $\pm$ are the same sign as the one immediately preceding each group, we shall have

$$
\begin{array}{l}
+ \cdots + - \cdots - + \cdots + - \cdots - + \cdots + \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + - \\
\hline
+ \cdots + - \cdots - + \cdots + - \cdots - + \cdots + \\
\quad - \cdots - + \cdots + - \cdots - + \cdots + - \cdots - \\
\hline
+ \pm \cdots - \pm \cdots + \pm \cdots - \pm \cdots + \pm \cdots -
\end{array}
$$

This is the smallest possible number of variations in the product and is exactly one greater than the number of variations in $Q(x)$. Hence multiplying by the m factors $x - r_1$, $x - r_2$, etc., will introduce into the product at least m variations in signs.

Note that the number of variations of signs in $f(x)$ may be much greater than m. Any one multiplication may introduce more than one new variation, and besides, there may be variations in $Q(x)$.

If any one of the ambiguous signs in the above multiplication is given a sign different from that which we gave it to get the smallest possible number of variations in the result, then the number of variations in the result will either be left unchanged or will be increased by two. Hence each multiplication by $x - r_i$ will increase the number of variations by an odd number. This leads to the following modified statement of Descartes' rule.

If in $f(x) = 0$ *there are* k *variations in sign then the number of positive roots is* k *less an even number. (Zero is here regarded as an even number.)*

EXERCISES

1. How may Descartes' rule of signs be used to determine a maximum number of negative roots in an equation?

2. What is the maximum number of positive roots in $x^3 - 5x^2 - 4x - 6 = 0$? What is the maximum number of negative roots in this equation? What transformation must be made in order to answer this last question?

3. What is the maximum number of real roots in $x^3 - 5x^2 - 10 = 0$? What is the maximum number of negative roots in this equation? How many complex roots does this equation have?

4. Exactly how many complex roots does the equation $x^3 + 2x^2 + 6 = 0$ have?

5. What are the roots of $(x^2 + x + 1)(x^2 - 2x + 2) = 0$? Multiply the factors in the left member of this equation and apply Descartes' rule of signs. Discuss this example.

236. Relations between roots and coefficients.—In studying quadratic equations we found that if r_1 and r_2 are the roots of $x^2 + p_1x + p_2 = 0$, then $r_1 + r_2 = -p_1$, and $r_1r_2 = p_2$. This turns out to be a special case of a general set of relations between the coefficients and the roots of rational integral equations.

$$x^2 + p_1x + p_2 = 0$$
$$r_1 + r_2 = -p_1 \qquad r_1r_2 = p_2$$

Example. If r_1, r_2, r_3 are the roots of $x^3 + p_1x^2 + p_2x + p_3 = 0$, then $(x - r_1)(x - r_2)(x - r_3) = 0$ is identical with the given equation. By multiplying, the left member reduces to $x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$. Hence we have the relations given above at the right.

$$x^3 + p_1x^2 + p_2x + p_3 = 0$$
$$r_1 + r_2 + r_3 = -p_1$$
$$r_1r_2 + r_1r_3 + r_2r_3 = p_2$$
$$r_1r_2r_3 = -p_3$$

If r_1, r_2, r_3, r_4 are the roots of $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$, then we can prove in exactly the same way the relations at the right.

$$r_1 + r_2 + r_3 + r_4 = -p_1$$
$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = p_2$$
$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -p_3$$
$$r_1r_2r_3r_4 = p_4$$

Carry out the indicated multiplication $(x - r_1)(x - r_2)(x - r_3)(x - r_4)$ to show that the above statement is correct.

The symbol Σ (Greek, Sigma) is often used to indicate sum. Thus Σr_i represents the sum $r_1 + r_2 + r_3$ or $r_1 + r_2 + r_3 + r_4 + \ldots$, depending on the degree of the equation that is involved; and Σr_ir_j represents the sum of all products of the type r_1r_2, r_1r_3, r_2r_3, ... Σr_i means the sum of all r's, and Σr_ir_j means the sum of all products of roots taken two at a time. $\Sigma r_ir_jr_k$ represents the sum of the products of the roots when these are taken three at a time, and so on.

Using this notation and the equation $x^n + p_1x^{n-1} + \ldots + p_n = 0$ whose roots are $r_1, r_2, \ldots r_n$, we have $\Sigma r_i = -p_1$, $\Sigma r_ir_j = p_2$, $\Sigma r_ir_jr_k = -p_3, \ldots$, $r_1r_2r_3 \cdots r_n = (-1)^n p_n$.

That is, if in an equation of the nth degree the coefficient of x^n is unity, then the coefficient of x^{n-1} is minus the sum of the roots, the coefficient of x^{n-2} is the sum of the products of the roots taken two at a time, the coefficient of x^{n-3} is minus the sum of the products of the roots taken three at a time, and so on to the absolute term, a_n, which is plus the product of all the roots if n is even and minus this product if n is odd.

The expressions, Σr_i, $\Sigma r_j r_i$, $\Sigma r_i r_j r_k$, . . . are all called symmetric functions of the roots of the equation.

In general an expression is a symmetric function of a set of quantities if any two of the quantities may be interchanged without changing the value of the expression.

Obviously there are many symmetric functions of the roots of an equation besides those given here. Thus Σr_i^n is a symmetric function of the roots for all values of n.

It may appear that these relations will enable us to solve any equation. This, however, is not the case. An example will illustrate:

Consider the equation $x^3 - 4x^2 + 7x - 5 = 0$. Let the roots be r_1, r_2, r_3. Then we have the relations at the right. Multiply (1) by r_1^2, (2) by $-r_1$, and (3) by 1, and add. Then $r_1^3 = 4r_1^2 - 7r_1 + 5$, or $r_1^3 - 4r_1^2 + 7r_1 - 5 = 0$, which is exactly the equation with which we started.

$$r_1 + r_2 + r_3 = 4 \quad (1)$$
$$r_1r_2 + r_1r_3 + r_2r_3 = 7 \quad (2)$$
$$r_1r_2r_3 = 5 \quad (3)$$

These three equations in r_1, r_2, r_3 constitute essentially a third degree problem, which leads to a third degree equation no matter how the three given equations are manipulated.

If, however, we have one or more relations among the roots besides the equation itself, then in some cases we may use the symmetric functions to solve the equations.

Example 1. If two of the roots of $x^3 + p_1x^2 + p_2x + p_3 = 0$ are equal, such as $r_1 = r_2$, then $r_1 + r_2 + r_3 = 2r_1 + r_3 = -p_1$ and $r_1r_2 + r_1r_3 + r_2r_3 = r_1^2 + 2r_1r_3 = p_2$. On substituting $r_3 = -p_1 - 2r_1$ in (2), we have a quadratic in r_1, and hence we can find both r_1 and r_2.

$$2r_1 + r_3 = -p_1 \quad (1)$$
$$r_1^2 + 2r_1r_3 = p_2 \quad (2)$$

Example 2. One of the roots of $x^3 + p_1x^2 + p_2x + p_3 = 0$ is twice one of the other roots. Find the three roots of the equation.

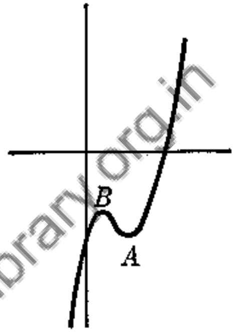
The roots are $r_1, 2r_1, r_2$. From the equations at the right we may eliminate r_2 and solve for r_1, finding its values to be $\dfrac{-3p_1 \pm \sqrt{9p_1^2 + 28p_2}}{28}$. Check this solution. Find the value of r_2 for each of these values of r_1.

$$3r_1 + r_2 = -p_1$$
$$2r_1^2 + 3r_1r_2 = p_2$$

Explain the two values of r_1. For what relation between p_1 and p_2 are these values complex?

237. ***Maxima and minima.***—As an interesting application of symmetric functions we shall consider the problem of finding maxima and minima of integral rational algebraic functions.

Example 1. Find a maximum and a minimum of the function $x^3 - 2x^2 + x - 3$.



Solution: We know that the graph of $y = x^3 - 2x^2 + x - 3$ is of the general form shown in the figure at the right. It is evident that A is a minimum point and B a maximum point, and our problem is to find the coordinates of these points.

In general, if the equations $y = b$ and $y = x^3 - 2x^2 + x - 3$ are solved simultaneously, that is, if we solve $x^3 - 2x^2 + x - 3 - b = 0$, there will be three values of x, and if these are all real they will represent the x-coordinates of the three points in which the line $y = b$ meets the curve. If the line $y = b$ passes through the point A or through the point B it will meet the curve in only two distinct points. In that case two of the roots of $x^3 - 2x^2 + x - 3 - b = 0$ will be coincident.

Let the roots be r_1, r_2, r_3, and let $r_1 = r_2$.

Then from the theorem on symmetric functions we know that

$$r_1 + r_1 + r_3 = 2,\ r_1r_1 + r_1r_3 + r_1r_3 = 1,\ r_1r_1r_3 = 3 + b,$$

or

$$2r_1 + r_3 = 2,\ \ r_1^2 + 2r_1r_3 = 1,\ \ r_1^2r_3 = 3 + b.$$

The first two of these equations are sufficient to determine r_1 and r_3. Substituting for r_3 in the second equation we have

$$r_1^2 + 2r_1(2 - 2r_1) = 1.$$

Solving for r_1 gives

$$r_1 = 1/3,\ r_1 = 1.$$

Hence the x-coordinate of the point A is 1 and of B is $1/3$.

To find the minimum value of the functions substitute $x = 1$ obtaining -3.

To find the maximum substitute $x = 1/3$ obtaining $-2\,{}^{23}/_{27}$.

By this method the maxima and minima of any cubic function may be found by solving a quadratic. In a similar way maxima and minima of a fourth degree function may be found by solving a cubic, those of a fifth degree function by solving a biquadratic, and so on.

Example 2. A rectangular piece of sheet metal is made into an open box by cutting out equal squares from the corners and turning up the four sides. Find the dimensions of the box of maximum volume which can be made in this way from a sheet 10 inches wide and 16 inches long.

SOLUTION: Let y be the volume of the box and x a side of the square to be cut out.

Then
$$y = x(10 - 2x)(16 - 2x)$$
$$= 4(x^3 - 13x^2 + 40x).$$

Clearly the maximum of the product $4(x^3 - 13x^2 + 40x)$ will be found if we find the maximum of the factor $x^3 - 13x^2 + 40x$.

Proceeding as in Example 1, we have

$$2r_1 + r_3 = 13, \; r_1^2 + 2r_1r_3 = 40.$$

Substituting for r_3,

$$3r_1^2 - 26r_1 + 40 = 0$$

and $$r_1 = \frac{26 \pm \sqrt{26^2 - 480}}{6} = \frac{26 \pm 14}{6} = 6\tfrac{2}{3} \text{ or } 2.$$

Since $6\frac{2}{3}$ cannot apply to this problem, it follows that $x = 2$ is the side of the square which gives the maximum volume.

Problems of this type are ordinarily solved by means of the calculus. However, the calculus reduces the problems to exactly the same algebraic equation that we obtain by the algebraic method. But the calculus enables us to write at once the final equations such as $r_1 + 2r_1(2 - 2r_1) = 1$ in Example 1 and $3r_1^2 - 26r_1 + 40 = 0$ in Example 2. The calculus may also be used for fractional and irrational functions to which the present method is not applicable.

EXERCISES

1. Find the maximum and minimum points on the curve

$$y = x^3 - 6x^2 + 4x - 12.$$

2. Find the maximum and minimum points on the curve

$$y = -x^3 + 10x^2 - 6.$$

3. Find the lowest point on the curve

$$y = x^4 - 6x^2 + 3x - 6.$$

SUGGESTION: Sketch the graph; then derive the necessary cubic equation and approximate the required root by using Horner's method.

EXERCISES AND PROBLEMS

By the use of Descartes' rule of signs find the maximum number of positive and of negative roots and any possible information about the number of imaginary roots of the following equations.

1. $2x^3 - x^2 + 3x - 5 = 0$
2. $x^3 + 5x^2 - 7x + 2 = 0$
3. $3x^3 + x^2 + x + 14 = 0$
4. $5x^3 - 2x^2 - 3x - 8 = 0$
5. $4x^3 + 12x^2 + 15 = 0$
6. $4x^4 - 3x^3 + 2x - 6 = 0$

Approximate to three decimal places the greatest real root of each of the following.

7. $x^3 - 9x - 28 = 0$
8. $x^3 + 9x^2 + 132x + 124 = 0$
9. $x^4 - 15x^2 - 20x - 6 = 0$
10. $x^4 - 7x^3 + 3x - 4 = 0$

11. Find the value of d in $x^3 + 2x^2 - 5x + d = 0$ if 2 and 3 are roots.

12. Find the values of c and d in $2x^3 - 10x^2 + cx + d = 0$ if 4 is a root and the difference of the other roots is 3.

13. Solve the equation $3x^3 - 7x^2 - 7x + 3 = 0$, having given that one root is the reciprocal of the other.

14. Solve the equation $2x^3 + x^2 - 5x + 2 = 0$ if one root is twice another.

15. Determine k so that one root of $x^3 + x^2 - 10x + k = 0$ will be twice another.

16. Determine k so that two of the roots of $x^3 + 4x^2 + kx - 18 = 0$ are equal.

17. Write the equation whose roots are -2, 1, 3, 4.

18. Determine the roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$, if one of the roots is known to be three times another root.

19. Form an equation with the roots -2, 4, 7.

20. Form an equation with the roots a, $2a$, $4a$.

21. Form an equation of which 3, 1, $1 + i$ are roots.

22. The product of three successive odd integers is 693. Find the integers.

23. The length of a box is 2 inches more than its depth, and its depth is 3 inches more than its width. What are the dimensions of the box if it contains 1950 cubic inches?

CHAPTER 19:

SOLUTION OF THE CUBIC AND THE BIQUADRATIC

In Chapter 17 a method was developed by means of which real roots of any algebraic equation with real coefficients may be approximated to any desired degree of closeness.

In this chapter general formulas will be developed giving all roots of the cubic and quartic (biquadratic).

238. *Reducing the general cubic to the form* $y^3 + py + q = 0$.—The general equation

$$x^3 + bx^2 + cx + d = 0 \tag{1}$$

may be reduced to the form

$$y^3 + py + q = 0 \tag{2}$$

by a substitution of the form $x = y + a$ and by giving a value to a which will make the coefficient of y^2 equal to zero.

It is found that this value of a is $-b/3$. Then substituting $x = y - b/3$ and reducing we have

$$y^3 + \left(c - \frac{b^2}{3}\right)y + d - \frac{bc}{3} + \frac{2b^3}{27} = 0$$

or, letting $c - \dfrac{b^2}{3} = p$ and $d - \dfrac{bc}{3} + \dfrac{2b^3}{27} = q$,

$$y^3 + py + q = 0. \tag{3}$$

If y_1, y_2, y_3 are the roots of (3), then

$$y_1 - \frac{b}{3}, y_2 - \frac{b}{3}, y_3 - \frac{b}{3} \text{ are the roots of (1).}$$

EXERCISE

Substitute $x = y + a$ in $x^3 + bx^2 + cx + d = 0$ and find the value of a which makes the coefficient of y^2 equal to zero. Using this value of a carry out completely the steps in the reduction outlined above.

239. Solving $y^3 + py + q = 0$ by reducing it to a quadratic in u^3.—In the equation

$$y^3 + py + q = 0 \tag{1}$$

substitute $y = u + v$.
Then by reducing,

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0. \tag{2}$$

Since u and v are subject only to the condition that their sum is a root of (1), we may impose the additional condition

$$3uv + p = 0, \text{ or } v = -\frac{p}{3u}. \tag{3}$$

Then (2) reduces to

$$u^6 + qu^3 - \frac{1}{27}p^3 = 0. \tag{4}$$

Hence,

$$u^3 = -\frac{q}{2} \pm \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}. \tag{5}$$

Denote one of the cube roots of $-\frac{q}{2} + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ by A and one of the cube roots of $-\frac{q}{2} - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ by B. Then $A, A\omega, A\omega^2, B, B\omega, B\omega^2$, are roots of (4) (see page 149).

We now note that $A \cdot B = -\frac{p}{3}$, for

$$\left(-\frac{q}{2} + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}\right)^{1/3} \left(-\frac{q}{2} - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}\right)^{1/3}$$
$$= \left(\frac{q^2}{4} - \frac{1}{4}q^2 - \frac{1}{27}p^3\right)^{1/3} = \left(-\frac{1}{27}p^3\right)^{1/3} = -\frac{1}{3}p.$$

Hence from (3), if $u = A$, then $v = B$,

and $$y = u + v = A + B \text{ is a root of (1).}$$

Similarly, $A\omega \cdot B\omega^2 = AB = -\frac{p}{3}$ and $A\omega^2 \cdot B\omega = AB = -\frac{p}{3}$.

Hence $y = A\omega + B\omega^2$ and $y = A\omega^2 + B\omega$ are roots of (1) and the equation is solved completely.

Suppose $\frac{1}{4}q^2 + \frac{1}{27}p^3$ is positive, so that the value of u^3 in (5) is real, then the corresponding value of v is real. Let A and B represent these values. Then the solution $y = A + B$ is real, while the solutions $y = A\omega + B\omega^2$ and $y = A\omega^2 + B\omega$ are both complex. This is the case when the cubic has one real and two complex roots.

Example 1. Solve the cubic $y^3 + 3y + 2 = 0$.

SOLUTION: Since $p = 3$ and $q = 2$, $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3} = \sqrt{2}$.

Hence $A = \sqrt[3]{-1 + \sqrt{2}} = .75$ and $B = \sqrt[3]{-1 - \sqrt{2}} = -1.34$ and $A + B = -.59$, $A\omega + B\omega^2 = .75\omega - 1.34\omega^2$, $A\omega^2 + B\omega = .75\omega^2 - 1.34\omega$ are the roots in which the A and B are approximated to two places of decimals.

Reduced to the form $a + bi$ these are: $-.59$, $.30 + 1.81i$, $.30 - 1.81i$.

Example 2. Solve $y^3 + 6y^2 + 21y + 20 = 0$.

SOLUTION: Substituting $y = x - 2$ (see §238) the equation reduces to

$$x^3 + 9x - 6 = 0. \tag{1}$$

Then $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3} = \sqrt{9 + 27} = 6$.

Hence $A = \sqrt[3]{3 + 6} = \sqrt[3]{9}$ and $B = \sqrt[3]{3 - 6} = \sqrt[3]{-3}$.

Hence $\sqrt[3]{9} + \sqrt[3]{-3}$, $\sqrt[3]{9}\omega + \sqrt[3]{-3}\omega^2$, $\sqrt[3]{9}\omega^2 + \sqrt[3]{-3}\omega$ are the roots of (1).

But $y = x - 2$ and the roots of the given equation are therefore

$$\sqrt[3]{9} + \sqrt[3]{-3} - 2,\ \sqrt[3]{9}\omega + \sqrt[3]{-3}\omega^2 - 2,\ \sqrt[3]{9}\omega^2 + \sqrt[3]{-3}\omega - 2.$$

Reduced to the form $a + bi$ these are: -1.36, $-2.32 + 3.05i$, $-2.32 - 3.05i$.

EXERCISES

1. If $\omega = \dfrac{-1 + \sqrt{3}i}{2}$, show that $\omega^2 = \dfrac{-1 - \sqrt{3}i}{2}$. See page 149, §153.

Find the roots of the following.

2. $x^3 + 3x^2 + 4x + 4 = 0$.
3. $x^3 - 6x^2 + 15x - 12 = 0$.
4. $x^3 - 3x^2 - 6x - 20 = 0$.
5. $x^3 + 9x^2 - 10x + 16 = 0$.

6. Is it possible to make a substitution of the form $y = x + a$ in the equation $y^3 + by^2 + cy + d = 0$ so as to reduce it to the form $x^3 + px^2 + q = 0$? If so, find the values of a that will make the coefficient of x equal to zero. What kind of equation in a must you solve?

7. Is it possible to make a substitution of the form $y = x + a$ in $y^3 + by^2 + cy + d = 0$ so as to reduce it to the form $x^3 + px^2 + qx = 0$? What equation in a would you have to solve? If this reduction were possible, could the cubic be solved?

240. *The irreducible case in the solution of the cubic.*—In case the cubic has three real roots, it turns out that $\frac{1}{4}q^2 + \frac{1}{27}p^3$ is negative and hence the value of u^3 obtained in §239 is complex. Since there is no convenient algebraic method for finding a cube root of a complex number, the above solution does not serve to evaluate the roots in case they are all real. This is called the irreducible case.

It is easily shown that in the irreducible case all the roots of the cubic are real:

Let $R = -\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)$. Then R is positive and

$$-\frac{q}{2} + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3} = -\frac{q}{2} + \sqrt{R}i.$$

Then $A = \sqrt[3]{-\frac{q}{2} + \sqrt{R}i}$ and $B = \sqrt[3]{-\frac{q}{2} - \sqrt{R}i}$.

We know that $-\frac{q}{2} + \sqrt{R}i$ has a cube root of the form $a + bi$ where a and b are real numbers.

Then $(a + bi)^3 = -\frac{q}{2} + \sqrt{R}i$.

Expanding and equating real and imaginary parts it follows that $a - bi$ is a cube root of $-\frac{q}{2} - \sqrt{R}i$.

Noting that $\omega + \omega^2 = -\frac{1 - \sqrt{3}i}{2} + \frac{-1 - \sqrt{3}i}{2} = -1$

and $\omega - \omega^2 = -\frac{1 - \sqrt{3}i}{2} + \frac{1 + \sqrt{3}i}{2} = \sqrt{3}i,$

then

$$A + B = a + bi + (a - bi) = 2a$$
$$A\omega + B\omega^2 = (a + bi)\omega + (a - bi)\omega^2 = -a - b\sqrt{3}$$
$$A\omega^2 + B\omega = (a + bi)\omega^2 + (a - bi)\omega = -a + b\sqrt{3}$$

which are the three real roots.

In the irreducible case the roots may be found by using trigonometric functions.

The next page, which deals with the case where all the roots of the cubic are real, should be studied by those only who have had a course in trigonometry in which complex numbers are represented in trigonometric functions.

241. Finding roots of the cubic in the irreducible case.—The expression $u^3 = -\frac{q}{2} + \sqrt{R}i$ may be represented by $r(\cos\theta + i\sin\theta)$

where $r = \sqrt{\frac{q^2}{4} + R}$ and $\theta = \tan^{-1}\left(-\frac{2\sqrt{R}}{q}\right)$.

Then $A = \sqrt[3]{r}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$ and $B = \sqrt[3]{r}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right)$.

Hence
$$A + B = 2\sqrt[3]{r}\cdot\cos\frac{\theta}{3}.$$

$$A\omega + B\omega^2 = \sqrt[3]{r}\left[\cos\frac{\theta}{3}(\omega + \omega^2) + i\sin\frac{\theta}{3}(\omega - \omega^2)\right]$$
$$= \sqrt[3]{r}\left[-\cos\frac{\theta}{3} - \sqrt{3}\sin\frac{\theta}{3}\right]$$
$$A\omega^2 + B\omega = \sqrt[3]{r}\left[\cos\frac{\theta}{3}(\omega + \omega^2) + i\sin\frac{\theta}{3}(\omega^2 - \omega)\right]$$
$$= \sqrt[3]{r}\left[-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right].$$

The numerical values of these roots may now be found by using a table of natural trigonometric functions.

Example. Solve the cubic $x^3 - 6x + 2 = 0$.

SOLUTION: In this case $A = (-1 + \sqrt{7}i)^{1/3}$, $B = (-1 - \sqrt{7}i)^{1/3}$ and the roots are: $(-1+\sqrt{7}i)^{1/3} + (-1-\sqrt{7}i)^{1/3}$, $(-1+\sqrt{7}i)^{1/3}\omega + (-1-\sqrt{7}i)^{1/3}\omega^2$, and $(-1 + \sqrt{7}i)^{1/3}\omega^2 + (-1 - \sqrt{7}i)^{1/3}\omega$.

If $\theta = \tan^{-1}(-\sqrt{7})$, then $-1 + \sqrt{7}i = 2\sqrt{2}(\cos\theta + i\sin\theta)$ and $-1 - \sqrt{7}i = 2\sqrt{2}(\cos\theta - i\sin\theta) = 2\sqrt{2}[\cos(-\theta) + i\sin(-\theta)]$.

Substituting these values in the roots and noting that

$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, we have

$$(-1 + \sqrt{7}i)^{1/3} + (-1 - \sqrt{7}i)^{1/3} = 2\sqrt{2}\cos\frac{\theta}{3}$$
$$(-1 + \sqrt{7}i)^{1/3}\omega + (-1 - \sqrt{7}i)^{1/3}\omega^2 = 2\sqrt{2}\cos\frac{\theta + 2\pi}{3}$$
$$(-1 + \sqrt{7}i)^{1/3}\omega^2 + (-1 - \sqrt{7}i)^{1/3}\omega = 2\sqrt{2}\cos\frac{\theta + 4\pi}{3}.$$

By using a table of trigonometric functions the roots may be reduced to decimal approximations.

242. *Solution of the biquadratic.*—The general fourth degree equation may be reduced to the form

$$x^4 + 2px^3 + qx^2 + 2rx + s = 0. \tag{1}$$

This equation is called the biquadratic and also the quartic. The factor 2 is introduced into the second and fourth terms to avoid fractions in the solution.

The method of solution which we shall use depends upon the possibility of adding an expression to both members of the equation so that the left member shall be the square of a second degree expression in x and so that the right member shall be the square of a first degree expression in x. If this can be done, we can take the square roots of both members and the solving of the quartic will be reduced to the solving of two ordinary quadratics.

Transposing,

$$x^4 + 2px^3 = -qx^2 - 2rx - s. \tag{2}$$

We now add an expression to both members so as to make the left member a square of the type $(x^2 + px + k)^2$.

Since $\quad (x^2 + px + k)^2 = x^4 + 2px^3 + (p^2 + 2k)x^2 + 2pkx + k^2$

it follows that $(p^2 + 2k)x^2 + 2pkx + k^2$ must be added to both members of (2). This gives

$$(x^2 + px + k)^2 = (p^2 + 2k - q)x^2 + (2pk - 2r)x + k^2 - s. \tag{3}$$

Since the left member is a square for all values of k, it remains to find a value of k which will make the right member a square. By §152 this requires that,

$$(2pk - 2r)^2 - 4(p^2 + 2k - q)(k^2 - s) = 0$$

or, $$2k^3 - qk^2 + 2(pr - s)k + sq - r^2 - sp^2 = 0. \tag{4}$$

This is a cubic in k. Let k_1, k_2, k_3, be its roots. If any one of these roots, as k_1, is substituted in (3) it will be reduced to

$$(x^2 + px + k_1)^2 = (m_1x + n_1)^2.$$

Hence, $$x^2 + px + k_1 = m_1x + n_1 \tag{5}$$

and, $$x^2 + px + k_1 = -m_1x - n_1. \tag{6}$$

The four roots, x_1, x_2, x_3, x_4 of (5) and (6) are the roots of (1).

Since the biquadratic cannot have more than four roots it follows that using k_2 or k_3 must give the same roots. That is, the four roots obtained from

$$x^2 + px + k_2 = m_2x + n_2 \tag{7}$$
$$x^2 + px + k_2 = -m_2x - n_2. \tag{8}$$

will be the same as those obtained from (5) and (6) as will also the roots obtained from

$$x^2 + px + k_3 = m_3x + n_3 \tag{9}$$
$$x^2 + px + k_3 = -m_3x - n_3 \tag{10}$$

The roots x_1, x_2, x_3, x_4 will however be differently paired in the different sets.

Thus if x_1, x_2 are the roots of (5) and x_3, x_4 are the roots of (6), then x_1, x_3 may be the roots of (7), and if so x_2, x_4 are the roots of (8); again x_1, x_4 may be the roots of (9) and if so, x_2, x_3 are the roots of (10).

This solution of the biquadratic depends upon the solution of the cubic (4). This cubic is called the resolvent cubic. Many methods have been devised for solving the biquadratic, but they all have this in common, that they involve the solution of a cubic.

The solution of the cubic and biquadratic given in this chapter are not in general adapted to finding the numerical values of real roots. For this purpose Horner's method or a similar one is always used. The principal interest in these solutions lies in the proof which they afford that the solutions of the cubic and biquadratic can be obtained by a finite number of steps involving the operations of addition, subtraction, multiplication, division, and the extraction of square and cube roots.

The question naturally arises as to whether the general solution of the fifth degree equation can be obtained, for instance by making use of the solution of the biquadratic. This we now know cannot be done.

EXERCISE

In the equation $x^4 + 4x^3 - 3x^2 - 38x - 48 = 0$, what are the values of p, q, r, and s in the form in which equation (1), page 258, is written? Using the equation just given, obtain the forms (2)–(10) given above.

EXERCISES

1. By making use of equations obtained in the solution of the biquadratic (§242) show that the complex roots of the biquadratic are pairs of conjugate numbers.

SUGGESTIONS: Prove the following statements:

(a) The roots of a quadratic equation with real coefficients, if complex, form a conjugate pair.

(b) If k_1 is a real root of the resolvent cubic, then

$$x^2 + px + k_1 = m_1x + n_1$$
$$x^2 + px + k_1 = -m_1x - n_1,$$

are quadratics with real coefficients.

Note that the resolvent cubic always has one real root.

2. Find the three roots of $x^3 + 3x - 4 = 0$, giving all real numbers correct to two places of decimals.

3. Using a table of natural trigonometric functions find the roots of $x^3 - 8x - 6 = 0$ correct to three places of decimals.

4. Reduce $x^3 + 6x^2 - 7x + 8 = 0$ to the form $x^3 + px + q = 0$ and decide without solving whether the equation has one or three real roots.

5. Reduce $x^4 + 2x^3 + x^2 + 4x - 10 = 0$ to the general form $(x^2 + 2px + k)^2 = (mx + n)^2$.

Find the resolvent cubic in k and without solving this equation decide whether it has one or three real roots.

By using a table of trigonometric functions the roots may be reduced to decimal approximations.

6. Show that in case all roots of a cubic are real and one of the roots is $2\sqrt[3]{r}\cos\frac{\theta}{3}$, then the other two roots are $2\sqrt[3]{r}\cos\frac{\theta + 2\pi}{3}$ and $2\sqrt[3]{r}\cos\frac{\theta + 4\pi}{3}$.

7. Find a method for reducing the general biquadratic to the form $x^4 + px^2 + qx + s = 0$.

SUGGESTION: Substitute $x = y + h$ in the general equation and find the value of h which will make the coefficient of x^3 equal to zero.

Attempt to solve this equation, $x^4 + px^2 + qx + s = 0$, by assuming $x^4 + px^2 + qx + s = (x^2 + kx + l)(x^2 - kx + m)$.

Show that determining k, l, m by equating coefficients leads to a cubic in k^2. This is the resolvent cubic.

CHAPTER 20:

PERMUTATIONS; COMBINATIONS

In this chapter we shall develop certain concepts and theorems or formulas, by means of which some interesting problems may be solved, and we shall study a certain type of analysis that must often be used in applying the formulas in the solution of problems.

243. ***Combinations; permutations.***—Any collection or group of things is called a combination. A combination remains the same so long as the things or elements of which it is made up remain the same. The placing, arrangement, or mutual position, of these elements does not affect the combination. A permutation of the elements in a combination has reference to their arrangement. The letters *a*, *b*, *c* form *one* combination consisting of these letters, but they form the following six permutations

abc   *acb*   *bac*   *bca*   *cab*   *cba*

Again, from the letters *a*, *b*, *c*, *d* we form four combinations with three letters in each, but for each of these combinations there are, as above, six permutations. Hence in a group of four different objects there are twenty-four permutations when taken three at a time. If *n* different objects are placed in any way in *n* different places, then each way of placing them is a permutation. Thus in arranging *a*, *b*, *c* we have the different places, "first place," "second place," "third place."

Combinations
abc
abd
acd
bcd

EXERCISES

1. Write the letters *a*, *b*, *c*, *d* in 24 different orders.
2. Write as many 5-place numbers as you can, using the digits 1, 2, 3, 4, 5. Do not use a digit more than once in any number.
3. Write as many three-place numbers as you can, using the digits 4, 5, 6. Each digit may be repeated as often as possible in any one number.
4. How many four-place numbers can be written, using the digits 1, 2, 3, 4?

244. ***Independent events.***—Going from a town A to a town B one may take any one of 3 roads. Going by any one of these roads will be regarded as an event. Going from B to C one may take any one of 4 roads, going by any one of which will also be regarded as an event. If the selection of a particular road between A and B does not influence the selection of a road between B and C, we say that the selections are independent. It is clear that there are 3 possible events in going from A to B and 4 possible events in going from B to C, and that there are $3 \times 4 = 12$ possible combinations of these in going from A to C.

If there are m events and then n events all independent of the m events, then there are $m \times n$ ways of combining these events. If there are m, n, and p events, all independent, then there are $m \times n \times p$ ways of combining these events, and so on for any groups of independent events.

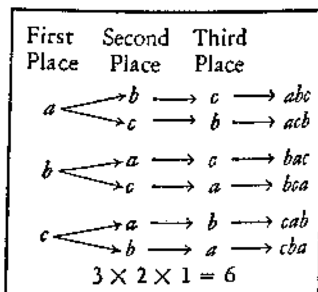
245. ***Definition of factorial.***—The product $1 \cdot 2 \cdot 3$ is called factorial 3, the product $1 \cdot 2 \cdot 3 \cdot 4$ is called factorial 4, and, in general, the product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ is called factorial n. Factorial n is written $n!$ and sometimes $\lfloor n$. To make certain formulas run smoothly we define factorial zero (0!) as equal to 1. It is convenient to note that $n! = n \cdot (n-1)!$.

$$1 \cdot 2 \cdot 3 = 3!$$
$$1 \cdot 2 \cdot 3 \cdot 4 = 4!$$
$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

This may be written $n(n-1)!$, if we remember that the factorial sign, !, applies to the last factor only.

246. ***The number of permutations of n things taken r at a time.***—We shall begin with a simple example.

Using the 3 letters a, b, c and the 3 places, first, second, and third, we see that we can fill the first place in 3 ways (with a, b, or c). There are 3 events. For each of these we can fill the second place in either one of two ways. These two events are independent of the first set of three events. Hence the first two places can be filled in $3 \times 2 = 6$ ways.



For each of these 3×2 ways of filling the first two places, the third places can be filled in only one way. Hence the three letters can be put in the three places in $3 \cdot 2 \cdot 1$ ways, that is, in $3!$ ways.

If we are to put the four letters a, b, c, d in four places, the first place can be filled in 4 ways; and for each of these ways of filling the first place, the second place can be filled in 3 ways; for each of these 4×3 ways of filling the first and second places, the third place can be filled in 2 ways; and for each of these, the fourth place can be filled in 1 way. That is, the four places can be filled in $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ ways.

In exactly this way we may show that 5 objects may be put in 5 places in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways, and in general that n objects may be put in n places in $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$ ways.

If we have 4 objects and only 2 places, then it is obvious that these places can be filled in $4 \cdot 3$ ways. That is, the first place can be filled in 4 ways, and the second in 3 ways, and that ends the process.

In general if there are r places and n objects, then the first place can be filled in n ways, the second in $n - 1$ ways, the third in $n - 2$ ways, and so on, to the rth place, which can be filled in $n - r + 1$ ways. That is, these r places can be filled in $n(n-1)(n-2) \ldots (n-r+1)$ ways.

But it is easily seen that

$$n(n-1)(n-2) \ldots (n-r+1) = \frac{n(n-1)(n-2) \ldots 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot \ldots (n-r)} = \frac{n!}{(n-r)!}$$

This formula gives the number of permutations of n things taken r at a time. This is denoted by $_nP_r$. The statement above shows that the number of permutations of n things taken n at a time, $_nP_n$, is $n!$

$$_nP_n = n! \qquad _nP_r = \frac{n!}{(n-r)!}$$

Thus, $$5 \cdot 4 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{5!}{3!}$$

and $$6 \cdot 5 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{6!}{4!}$$

EXERCISES

1. If $n = 6$ and $r = 3$, show that

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n(n-1)(n-2)\ldots 2\cdot 1}{1\cdot 2\cdot 3\cdot \ldots (n-r)} = \frac{n!}{(n-r)!}$$

2. Show the same result if $n = 6$, $r = 4$; if $n = 6$, $r = 5$.
3. If $n = 8$, $r = 5$, what is the value of ${}_nP_r$?

In solving problems involving permutations, we not only need to use the formulas just developed, but we also need to use certain ways of thinking that are, to an extent, peculiar to this subject and to that of the next chapter.

Example 1. How many odd three-figure numbers can be written using the digits 3, 4, 6, 7, 8, 9, no digit being used more than once in any one number?

SOLUTION: The figure to the right must be an odd number, and hence the first place at the right can be filled in 3 ways. For any one way of filling the place at the right, the next place can be filled in 5 ways, and then the third place in 4 ways. Hence the places can be filled in $3 \cdot 5 \cdot 4 = 60$ ways.

Example 2. Find the value of n if ${}_nP_6 = 8\,{}_nP_5$.

SOLUTION: ${}_nP_6 = \dfrac{n!}{(n-6)!}$, ${}_nP_5 = \dfrac{n!}{(n-5)!}$

Then $\dfrac{n!}{(n-6)!} = 8\,\dfrac{n!}{(n-5)!}$

In solving this equation note that $(n-5)! = (n-5)(n-6)!$. Then cancel $\dfrac{n!}{(n-6)!}$ in both members of the equation.

$$\frac{n!}{(n-6)!} = 8\,\frac{n!}{(n-5)!}$$

$$1 = \frac{8}{n-5}$$

$$n = 13$$

Example 3. How many four-figure numbers are divisible by 5?

SOLUTION: The number in ones' place must be a 0 or a 5. Hence this place may be filled in 2 ways. The tens' and hundreds' places may each be filled in any one of 10 ways, and the thousands' place may be filled in 9 ways. (Note that 0 cannot be used in this place.) Hence the required number is $2 \cdot 10 \cdot 10 \cdot 9 = 1800$.

PROBLEMS

1. Write all possible combinations of the letters a, b, c, d taken two at a time.
2. Write all possible numbers that can be formed from the three digits 1, 2, 3, no digit being used more than once in any number.
3. A man can leave a college building by any one of three doors and enter another building by four doors. In how many different ways can he leave the first building and enter the second?

4. The walls of a room may be painted any one of five colors and the ceiling any one of three colors. In how many different ways can the colors be chosen?

5. There are three roads from A to B, five roads from B to C, and 2 roads from C to D. In how many ways can a person travel from A to D?

6. If three students enter a classroom which has 20 chairs, in how many different ways can they be seated?

7. How many different numbers each having 3 digits can be formed from the digits 1, 3, 5, 7, and 9 without repetitions?

8. How many signals can be made with 5 flags of different colors using three at a time?

9. How many different permutations, each of 3 letters, can be made from the letters of the word *house*?

10. In how many ways may 5 people be seated in a six-passenger car? In how many ways may 6 people be seated in this car?

11. If ${}_nP_r = 8_{n-1}P_{r-1}$, find n.

12. If ${}_nP_5 = 12_nP_3$, find n.

13. If the number of permutations of n things four at a time is fourteen times the number of permutations of $n - 2$ things three at a time, find n.

14. Four persons enter a carriage in which there are six seats; in how many ways can they take their places?

15. How many different six-place numbers can be formed by using six out of the nine digits 1, 2, 3, . . . , 9 if no digit is repeated?

16. How many numbers of four different digits each can be formed from the digits 1, 2, 3, 4, 5, 6, and 8, if each number is to begin and end with an odd digit?

17. In how many ways can three positions be filled by selections from twelve people?

18. There are five applicants for two positions. In how many ways can they be filled?

19. In how many ways can a host seat five guests if seven seats are available?

20. How many permutations are there of the letters in *actions* taken (a) three at a time; (b) five at a time?

21. In how many ways can a man distribute a nickel, a dime, and a quarter among five boys?

22. Find an expression for the number of permutations of n different things taken four at a time.

23. There are four roads between cities A and B, and three roads between B and C. In how many ways can a person travel from A to B to C and return without passing over any road twice?

24. In how many ways can a certain three speakers be assigned positions on a program listing nine speakers?

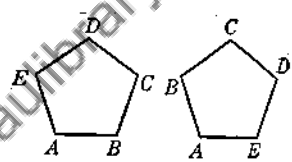
247. Permutations in cyclic order.—Objects arranged around a circle or any other simply closed curve are said to be in <u>cyclic order.</u>

Thus people seated around a table, the beads on a closed string, or the vertices on a polygon, are in cyclic order.

In such an arrangement there is no first or last place. While strictly speaking in such an arrangement any object is between any other two, we will speak of an object as between two others when it is immediately next to them.

Thus in a pentagon $ABCDE$, B is between A and C, C is between B and D, D is between C and E, E is between D and A, and A is between E and B.

The number of permutations in cyclic order depends upon whether or not opposite orders around the cycle are counted as different permutations. Thus the two arrangements of the letters shown in the pentagons $ABCDE$ differ only in the right and left (or clock-wise and counter clock-wise) order of the letters. In both, A is between E and B, B between A and C, etc. Unless otherwise specified, opposite orders of this kind will be regarded as different cyclic permutations.



In distinction from cyclic order, we speak of the order we have considered earlier as <u>linear order</u>.

Note that in linear order when starting from the first place, it is possible to go in only one "direction," while in cyclic order one may go in two directions, as to the left or to the right.

Problem. Find the number of cyclic permutations of n objects taken n at a time.

SOLUTION: It is evident that, where the n objects have been arranged in order around a cycle, each object may be moved one place to the right (or the left) without disturbing the order. Hence each object may occupy any one of n different places without changing the cyclic order. It follows that for each cyclic order there will be n linear orders. Therefore, the number of cyclic orders multiplied by n gives the number of permutations.

That is, the number of cyclic permutations is $\frac{{}_nP_n}{n} = \frac{n!}{n} = (n-1)!$

If opposite cyclic orders are regarded as the same cyclic permutation this number must be divided by two.

Example 1. In how many different orders may 10 people be seated around a table?

SOLUTION: The required number is $9! = 362,880$.

Example 2. In a company there are 5 men and 5 women. In how many different orders may these be seated around a table if each man is to be seated between two women?

SOLUTION: Suppose the women are seated first. They may be arranged in $4! = 24$ ways. For each way of seating the women there are 5 vacant chairs in which any permutation of the men results in a different cyclic order of the whole company. Hence the total number of cyclic arrangements is $24 \times {}_5P_5 = 24 \cdot 120 = 2880$.

EXERCISES

1. In how many ways can 7 men be seated around a circular table?
2. Ten children are playing a game in which all but one of them stand in a circle. In how many different orders can they be arranged?
3. How many different signals can be made by running up 1, 2, 3, or 4 flags on a vertical line, using 7 flags of different colors?

248. ***Permutations of things not all different.***—Let N denote the number of permutations of n things of which p are alike. Consider any one of these permutations. If the p things were unlike they could be arranged among themselves in $p!$ ways. Hence every permutation in N would give rise to $p!$ permutations if all the n things were unlike. Hence

$$N \cdot p! = n! \quad \text{and} \quad N = \frac{n!}{p!}.$$

EXERCISES

1. In how many different ways can the letters of the word *permutation* be arranged?
2. In a group of 12 things 5 are alike. In how many ways can these things be arranged?
3. On a baseball squad there are 5 pitchers, 3 catchers, 3 short stops, and 12 who can play each of the remaining six positions. In how many ways can a team of 9 players be selected from this squad?
4. Prove that if of n things p are alike and also q are alike, then the number of permutations is $\dfrac{n!}{p! \cdot q!}$.
5. Prove that if of n things p are alike, q are alike, and also r are alike, then the number of permutations is $\dfrac{n!}{p! \cdot q! \cdot r!}$.

249. ***Number of combinations of n things taken r at a time.***—Denote this number by ${}_nC_r$. Since in one combination of r objects there are $r!$ permutations, it follows that

$$ {}_nC_r \cdot r! = {}_nP_r, \text{ and hence, } {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}. $$

Since $0! = 1$ we find the value of ${}_nC_0$ by substituting $r = 0$ in

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \text{ obtaining, } {}_nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1. $$

The formula $0! = 1$ may appear artificial, but a little reflection will make it seem reasonable: "If we have n objects, in how many ways can we take none of them?" The answer is: in just one way.

The formula ${}_nC_r = {}_nC_{n-r}$ is easily proved. If a combination of r things is removed from a group of n things, then a combination of $n - r$ things is left. That is, there are exactly as many combinations of $n - r$ things as there are of r things.

Combinations may be used conveniently, for instance in studying the binomial coefficients (see page 294).

Example 1. A group of people consists of 8 men and 11 women. In how many ways may a committee be selected if it is to consist of 3 men and 4 women?

SOLUTION: The men may be selected in ${}_8C_3$ ways and the women in ${}_{11}C_4$ ways and hence the whole committee may be selected in ${}_8C_3 \times {}_{11}C_4$ ways. But ${}_8C_3 \times {}_{11}C_4 = 56 \cdot 330 = 18,480$.

Example 2. In how many ways may two tennis teams of 5 players on each be selected from a group of 15 players?

SOLUTION: One team of 5 players may be selected in ${}_{15}C_5 = \dfrac{15!}{5! \cdot 10!}$ ways, and for each of these selections a second team may be selected from the remaining 10 players in ${}_{10}C_5 = \dfrac{10!}{5! \cdot 5!}$ ways.

Hence two teams may be selected in

$$ \frac{15!}{5! \cdot 10!} \cdot \frac{10!}{5! \cdot 5!} = \frac{15!}{(5!)^3} = 756,756 \text{ ways.} $$

EXERCISES

1. Six ladies meet at a party, and each shakes hands with each of the others; how many hand-shakes are there?

2. There are ten points, no three of which lie in the same straight line; how many distinct lines can be drawn through these points if each line passes through two of the points?

3. Write a formula giving the number of lines determined by n points, no three of which lie in the same straight line. (A line passing through two points is said to be determined by these points.)

4. How many triangles are determined by 8 points, no three of which are in the same straight line?

5. Write a formula giving the number of triangles determined by n points, no three of which are in the same straight line.

6. How many tetrahedrons are determined by 10 points, no four of which are in the same plane?

(A tetrahedron is determined by its four vertices.)

7. Write a formula giving the number of tetrahedrons determined by n points, no four of which are in the same plane.

8. (A plane is determined by three points if they not all lie in one straight line.) How many planes are determined by 6 points no three of which lie in a straight line and no four of which lie in the same plane?

9. Make a formula showing how many planes are determined by n points, if no three lie in a straight line and no four lie in the same plane.

10. On a certain railway division there are 16 stations. How many different kinds of tickets are needed to provide tickets between any two of these stations if the same kind of ticket is good in either direction?

11. A party of 11 people go in two cars; in how many ways can the groups for the cars be selected if one car carries 6 people and the other 5 people?

12. A committee of 5 women and 4 men is to be selected from a group of 9 women and 7 men; in how many ways can the committee be selected?

13. In how many ways can an ace and king be selected from a pack of cards (a) if they are to be of the same suit? (b) if they are to be of different suits? (c) if there is no such restriction?

14. In how many ways can one hand of 13 cards be dealt from a pack of 52 cards?

15. In how many ways can the four hands of whist be dealt from a pack of 52 cards?

16. In a square arrangement of 9 letters arranged as at the right, how many groups of three letters can be arranged such that each group shall contain exactly one letter from each horizontal line and also exactly one letter from each vertical column?

a	*b*	*c*
d	*e*	*f*
g	*h*	*i*

17. Solve exercise 16 using a square containing 16 letters.

18. Solve the same problem using n^2 different objects arranged in a square as in exercises 16 and 17.

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*l*
m	*n*	*o*	*p*

The results in these exercises will be used in Chapter 25 when we come to study determinants.

MISCELLANEOUS PROBLEMS

The two groups of problems, A and B, are of about equal difficulty. Either group may be used.

Group A

1. In how many ways can three dimes and seven quarters be distributed among ten boys, each to receive a coin?

2. Twelve athletes compete in a race in which the first four places receive medals of the same kind. In how many ways can the medals be distributed?

3. From a group of seven friends, in how many ways can a man select a dinner party of four?

4. From five men and four women, in how many ways can we choose a committee of three men and two women?

5. The Greek alphabet has 24 letters. How many fraternity names, each of three letters, can be formed if repetitions are permitted? How many if repetitions are not permitted?

6. There are five roads between A and B. In how many ways can we go from A to B and return?

7. If the faces of a cubical die are numbered 1, 2, 3, 4, 5, and 6, respectively, and if two dice are tossed, in how many ways can a total of 7 show up on the dice? Name the dice A and B. If A shows 1 and B shows 6, and if A shows 6 and B shows 1, we have two different ways of showing 7.

8. In how many ways can five boys and three girls be seated in a row of eight seats with the girls side by side?

9. How many distinct signals can be made with twelve flags by displaying them all at a time on a vertical staff, if two flags are red, four are blue, three are white, and the rest are yellow?

10. In how many ways can six German books, seven French books, and four Italian books be arranged on a shelf so that all the books of each language are together?

11. In how many ways can we select groups of eight books, consisting of five French and three English books, from six different French and five different English books?

12. How many football teams consisting of 7 linemen and 4 backfield men can be selected from a team consisting of 25 candidates for the line and 18 for the backfield, it being immaterial as to which position a man plays in either the line or the backfield?

13. In how many ways can an arrangement of 4 letters be made out of the letters a, b, e, c, d, o, when there is no restriction as to the number of times a letter is repeated in each arrangement?

14. In how many ways can five different presents be distributed among four children?

15. From seven white and six black balls, in how many ways can we select four balls of one color?

16. In how many ways can ten people be divided among three hotels *A*, *B*, and *C*, three people in *A*, two in *B*, and five in *C*?

17. How many four-place numbers without repeated digits and greater than 3200 can be made by the use of (1, 2, 3, 0, 5)?

18. In how many different ways can six different books be arranged on a shelf with all books of the same color side by side, if two books are red and four are blue?

19. In how many ways is it possible to draw a sum of money from a bag containing a dollar, a half-dollar, a quarter, a dime, a five-cent piece, a two-cent piece, and a penny?

20. From five black and eight red balls, in how many ways can we select five balls so that (a) just two are black; (b) at least two are black?

21. How many committees of one or more can be made up of a group of eight people?

22. A boat's crew consists of eight men of whom one can row only on bow side and one only on stroke side; in how many ways can the crew be arranged?

23. By the use of six different red books and five different blue books, in how many ways can we exhibit six books in a row if we use four red and two blue books?

Group B

1. In how many different orders can six people take seats at a round table? In how many ways can they take seats in a row of six seats?

2. How many sums of money, each consisting of four coins, can be formed from a cent, a nickel, a dime, a quarter, and a dollar?

3. From five black and seven white balls, in how many ways can we select a set of five balls such that three are black and two are white?

4. In how many ways can nine books be given to two boys so that one will get five and another four?

5. In how many ways can a red book, a green book, and five different blue books be arranged together on a shelf with the red and green books separated?

6. In how many ways can ten books be arranged on a shelf so that three particular books will always be side by side?

7. How many signals can be made from four white flags, three green flags, and two red flags if all of them are arranged in line?

8. How many permutations can be made using all of the letters of the word *mathematics*?

9. How many telegraphic characters could be made by using three dots and two dashes in each character?

10. A signaling apparatus has five arms of different colors, and each arm has four distinct positions, including the position of rest; find the total number of signals that could be made.

11. From eight men and five women, in how many ways can we select a committee (a) of three men *and* three women; (b) of three men *or* three women?

12. In a baseball league of eight teams, how many games will be played if each team plays sixteen games with each of the other teams?

13. In how many ways can seven different presents be divided between *A* and *B*, three to *A* and four to *B*?

14. In how many ways can three books on history and five on mathematics be arranged on a shelf if books on the same subject stand together?

15. In how many ways can eight people be seated in a row of eight seats if a certain two (a) are to be side by side? (b) are not to be side by side?

16. By the use of 1, 2, 3, 4, 5, 6, 7, 8, how many numbers of five different digits each can be formed each consisting of three odd and two even digits? No number is to contain a repetition of a digit.

17. (a) In how many ways can eight persons form a ring? (b) Find the number of ways in which four gentlemen and four ladies can sit at a round table so that no two gentlemen sit together.

18. From seven men and four women, in how many ways can we select a committee of four involving (a) just two men; (b) at most two men?

19. One bag contains four white and six black balls, and a second bag contains five white and four black balls. In how many ways can we select three white and two black balls (a) if all come from the same bag? (b) if the white balls come from one bag and the black from the other? It is understood that the balls are numbered so they can be distinguished.

20. Five coins of different kinds are to be placed flat on a table. In how many ways can we arrange to have uppermost (a) five heads? (b) just two heads? (c) at least three tails?

21. A man is to play a game six times and we can arrange for him to win or to lose any game. In how many ways can we specify his wins and losses if he is to win (a) just two games? (b) at least four games?

22. If seven coins of different kinds are tossed together, in how many ways can they fall with (a) just four heads? (b) at least four heads?

23. By the use of *a*, *e*, *i*, *o*, *b*, *n*, *r*, *s*, *t*, how many permutations of six different letters each can be formed if each permutation consists of four consonants and two vowels?

CHAPTER 21:

PROBABILITY

The remark is often made that a certain event is "likely" to happen or that "probably" it will happen. We say that a certain event is more probable than another, or that two events are equally probable. A mathematical theory of probability is possible only when we assign a numerical value to the probability that a certain event will happen. Such a theory was developed early in connection with problems on gambling. What, for instance, is the probability that eleven points will turn up when two dice are thrown? The answer must be stated as a number.[1] Later the theory of probability assumed importance in connection with business undertakings such as insurance; still later it became important in the study of biology and in other sciences. In this chapter we shall study some of the more elementary problems involving probability.

250. Events that are equally probable.—If we know no reason why of two events one is more likely to occur than the other, we say that the events are equally probable. If a coin is thrown, we say it is equally probable to show head or tail. Again, if a die is thrown, we say it is equally probable to show 1, or 2, and so on up to 6.

251. Favorable and unfavorable events.—If we ask what the probability is that I will throw a head on flipping a coin, then throwing a head is called a favorable event, and throwing a tail is an unfavorable event. If we ask what the probability is that one or two will show on throwing a die, then there are two favorable events and four unfavorable, since the die may fall in any one of six ways, two of which are favorable. Whether an event is favorable depends upon which way the question is asked. If we ask

[1] This is a concrete example of the general fact that, with negligible exceptions, mathematics becomes applicable to a problem involving quantities only when these are expressed in terms of numbers.

about the probability that a certain man will die within a year, then his dying is a "favorable" event; but if we ask about the probability of his living a year, then his living is a favorable event.

252. Numerical value of a probability.—The probability that a coin will be thrown head up is $\frac{1}{2}$. There are two equally probable events, and one of them is favorable. The probability that an ace (one point) will show on throwing a die is $\frac{1}{6}$. There are six equally probable events, and one of them is favorable.

The probability that less than three points will be thrown on one throw of a die is $2/6 = 1/3$, since there are six equally probable events, and two of them are favorable.

In general, if there are $m + n$ equally probable events, and if m of these events are favorable and n are unfavorable, and if exactly one of these events will occur, then the probability is $\frac{m}{m+n}$ that this one event will be favorable and $\frac{n}{m+n}$ that it will be unfavorable. When we say that the odds are m to n that an event will occur we mean that the probability is $\frac{m}{m+n}$ that it will occur. When we throw a coin, the "odds are even" (equal) that it will show head or tail.

Problem. I take one ball out of a bag containing 4 red balls and 7 blue balls. What is the probability that I will take a red ball?

SOLUTION: It is assumed to be equally probable that any one of the 11 balls will be taken out. There are therefore 11 equally probable events and 4 favorable events. Hence the probability that I will take a red ball is 4/11.

EXERCISES

1. If one card is drawn at random from a pack of ordinary playing cards, what is the probability this card will be the ace of spades? What is the probability that it will be an ace?

2. If one card is drawn at random from a pack of cards, what is the probability that the card will be a spade? What is the probability that it will be black?

3. In a bag there are 9 red balls and 15 black balls. What is the probability that one ball drawn from this bag will be red? What is the probability that it will be black?

4. If a die is thrown, what is the probability that it will show less than 4 points? What is the probability that it will show more than 2 points?

5. If a letter is taken at random from the alphabet, what is the probability that it will be a vowel? a consonant?

253. *Mutually exclusive events.*—If of two events both cannot occur simultaneously, then these events are mutually exclusive. Thus, on flipping a coin, either head or tail may show, but not both. Either of these events "excludes" the other, and the events are mutually exclusive. In one situation there may be many events any two of which are mutually exclusive. Thus a thrown die may show 1 to 6 points, and any two of these are mutually exclusive.

The probability that a flipped coin will turn tail up is ½ and the probability that it will turn head is ½. The sum of these probabilities is 1.

$$\frac{1}{2} + \frac{1}{2} = 1$$
$$\frac{3}{7} + \frac{4}{7} = 1$$
$$\frac{2}{15} + \frac{7}{15} + \frac{6}{15} = 1$$

If a ball is taken from a bag containing 3 red balls and 4 blue balls, the probability that the ball is red is 3/7 and the probability that it is blue is 4/7. Again the sum of the probabilities is 1.

If a ball is taken from a bag containing 2 red balls, 7 yellow balls, and 6 black balls, the probabilities that it will be red, yellow, or black are respectively 2/15, 7/15, and 6/15. The sum of these probabilities is again 1.

These are examples of the following theorem.

If an event can occur in any one of several mutually exclusive ways, then the sum of the probabilities that it will occur in these different ways is unity.

It follows from this theorem that if the probability that one of two mutually exclusive events is p_1, then the probability that the other will occur is $1 - p_1$.

$$p + q = 1$$
$$q = 1 - p$$
$$p = 1 - q$$

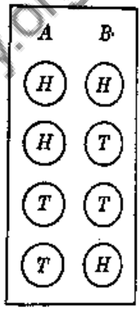
The probability that an event will be favorable is often denoted by p and that it will be unfavorable by q. Then the above theorem is: $p + q = 1$, whence $p = 1 - q$, and $q = 1 - p$.

EXERCISES

1. Verify that the above theorem holds in the examples on page 276.

2. If a letter is taken at random from the word *probability*, what is the probability that the letter will be a *b*? that it will be an *i*? that it will be a vowel? that it will be a consonant? Do these probabilities illustrate the above theorem? Explain fully.

254. *Composite events.*—What is regarded as an event may be a combination of two or more events. Such an event is called a composite event, the minor events of which it is composed being constituent events. Illustrations will make this idea clear: From a city *A* to a city *B* there are three roads labeled 1, 2, 3. From city *B* to a third city *C* there are four roads labeled *a*, *b*, *c*, *d*. Going from *A* to *C* by way of *B* one may take any of the three roads from *A* to *B* and then any of the four roads from *B* to *C*. Taking any combination such as roads 1 and *a* is a composite event, taking the road 1 from *A* to *B* and taking the road *a* from *B* to *C* being its constituent events.



Again in throwing two or more coins each event is composite. Consider two pennies, *A* and *B*. The event resulting when the two pennies are thrown is composite, consisting of the constituent event that results for each penny. The composite event may be two heads. This consists of the event head for each penny. The composite event two tails consists of the event tail for each penny. The composite event, one head and one tail, consists of the event head for *A* and the event tail for *B*, or of the event tail for *A* and the event head for *B*.

If three coins are thrown, we have composite events represented by *HHH*, *HHT*, *HTH*, *HTT*, *THH*, *THT*, *TTH*, *TTT*, eight in all.

255. *Probability of composite events.*—We shall study Examples:

Example 1. If in the above illustration, the roads 1, 2, 3 from *A* to *B* are equally popular and also the roads *a*, *b*, *c*, *d* from *B* to *C* are equally popular, what is the probability that a man going from *A* to *C* by way of *B* will take a given combination of roads, such as 1, *a*?

First solution: There are 12 possible combinations of roads: 1, *a*; 1, *b*; 1, *c*; 1, *d*; 2, *a*; 2, *b*; . . . , 3, *d*. That is, there are 12 equally probable events, while there is 1 favorable event, namely 1, *a*. Hence the required probability is 1/12.

Second solution: The probability that road 1 will be taken from *A* to *B* is 1/3, and the probability that road *a* will be taken from *B* to *C* is 1/4. The product of these probabilities, $1/3 \cdot 1/4 = 1/12$, is the probability that the composite event 1, *a* will occur.

Example 2. On flipping two pennies, what is the probability of each of the events, two heads, two tails, head and tail?

FIRST SOLUTION: There are four equally probable events shown in the figure on page 278. One of these events is two heads and one is two tails and hence the probability of each of these is 1/4. Each of the two events shown by *H*, *T* and *T*, *H* is an event showing one head and one tail. Hence the probability of the one-head—one-tail result is 2/4 or 1/2.

SECOND SOLUTION: The probability that each of the coins *A* and *B* will turn head is ½. Hence the probability that both will turn head is ½ · ½ = ¼.

The first solution in each of the above Examples depends upon the principle of §252, that the probability of a favorable event is m/n when the total number of events is n and the number of favorable events is m.

The second solution in each case really makes use of the following principle.

The probability of a composite event is equal to the product of the probabilities of its constituent events.

This evidently holds for any number of constituent events. That is, if $p_1, p_2, \ldots, p_n$ are the probabilities of each of the n events that are the components of a composite event, then $p_1p_2 \ldots p_n$ is the probability of the composite event.

The fact that in each of the above cases the two solutions are the same may be regarded as a verification of this general principle.

The truth of this theorem can be seen from the following example.

Suppose that in one bag we have m balls one of which is white, and in another n balls one of which is white. What is the probability that on taking one ball out of each bag both will be white?

SOLUTION: These balls can be drawn in mn different ways, one of these being the favorable event. Hence the probability of this event is $1/(m \cdot n) = p_1 \cdot p_2$. This can evidently be extended to any number of different bags.

EXERCISES

1. If one letter is taken at random from each of the words *number* and *position*, what is the probability that these letters will be *n* and *t*? What is the probability that they will be *n* and *o*? *b* and *i*?

2. If one letter is taken at random from each of the words *letter*, *common*, and *probability*, what is the probability that these letters will be *e*, *n*, and *b*? *t*, *m*, and *i*?

256. Typical problems in the probability of composite events.—Instead of giving formal rules for different types of problems and for different cases, we shall study a series of problems that come under the above general head.

Problem 1. In one of two bags there are 2 red balls and 3 black and in the other there are 5 red balls and 7 black. In taking one ball out of each bag what is the probability (a) that both will be red? (b) that both will be black? (c) that one will be red and one black?

Solution: The probability that a red ball will be taken out of the first bag is 2/5 and that a red ball will be taken out of the second is 5/12. Hence the probability that both balls will be red is $2/5 \cdot 5/12 = 1/6$. Similarly, the probability that both will be black is $3/5 \cdot 7/12 = 7/20$. Then the probability that one will be red and one black is $1 - 1/6 - 7/20 = 29/60$. Hence the required probabilities are 1/6, 7/20, and 29/60.

Problem 2. According to the American Experience Table of Mortality (see page 284) the probability that a man, age 30, will be alive 20 years from now is approximately 69/85, while the probability that his wife, age 25, will be alive 20 years from now is approximately 74/89. From these data find the probability of each of the following.

(a) That both will be alive 20 years from now.

(b) That both will be dead 20 years from now.

(c) That the husband will be dead and the wife alive 20 years from now.

(d) That the husband will be alive and the wife dead 20 years from now.

$$1 - \frac{69}{85} = \frac{16}{85} = .19$$

$$1 - \frac{74}{89} = \frac{15}{89} = .17$$

$$\text{(a)} \quad \frac{69}{85} \cdot \frac{74}{89} = \frac{51}{76} = .67$$

$$\text{(b)} \quad \frac{16}{85} \cdot \frac{15}{89} = \frac{4}{151} = .03$$

$$\text{(c)} \quad \frac{16}{85} \cdot \frac{74}{89} = \frac{8}{50} = .16$$

$$\text{(d)} \quad \frac{69}{85} \cdot \frac{15}{89} = \frac{23}{168} = .14$$

Suggestions for solution: Find the probability that husband will be dead in 20 years and also the wife. The results are given above at the right. What principle is used in finding these? The answers to (a), . . . , (d) are given approximately. Explain each of these. What principle is used?

Problem 3. In throwing two dice, what is the probability of throwing 7 points?

Solution: Denote the two dice by A and B. In throwing two dice there are 36 equally possible events, since A may fall in 6 ways and for each of these B may fall in 6 ways. But as shown at the right, there are 6 composite events that show a total of 7 points. Hence by §252, the probability of throwing 7 points is 1/6.

A	B
1	6
2	5
3	4
4	3
5	2
6	1

Problem 4. In throwing three dice, what is the probability of throwing 10 points?

SOLUTION: There are $6 \cdot 6 \cdot 6 = 216$ events, of which 27 are favorable. (See the table at the right.) Hence the required probability is $27/216 = 1/8$. The number of favorable events may be found without actually writing them down and counting. Thus a combination of 1, 3, and 6 may come about by taking permutations of these numbers, giving 6, 3, 1; 6, 1, 3; 3, 6, 1; 3, 1, 6; 1, 6, 3; 1, 3, 6; and similarly for the numbers 5, 4, 1; 5, 3, 2. Note that 2, 2, 6 can be arranged in only three ways as can also 4, 4, 2, and 3, 3, 4.

A	*B*	*C*
6	3	1
6	2	2
6	1	3
5	4	1
5	3	2
5	2	3
5	1	4
4	5	1
4	4	2
4	3	3
4	2	4
4	1	5
3	6	1
3	5	2
3	4	3
3	3	4
3	2	5
3	1	6
2	6	2
2	5	3
2	4	4
2	3	5
2	2	6
1	6	3
1	5	4
1	4	5
1	3	6

Problem 5. A bag contains 9 black balls and 5 red balls. What is the probability of drawing 4 black balls in succession?

SOLUTION: The probability of drawing one black ball is 9/14. This leaves 8 black balls and 5 red balls. The probability of drawing a black ball from this combination is 8/13. The probability of drawing another black ball is 7/12 and of drawing still another is 6/11. The product of these is the required probability.

$$\frac{9}{14} \cdot \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{18}{143}$$

EXERCISES

1. In throwing three dice, what is the probability that 8 points will be thrown? Make a table like the one at the right.

2. In throwing three dice, compare the problem of the probability that 4 and 17 points will be thrown. Find each of these probabilities.

3. In throwing three coins, what is the probability that each of the following combinations will be thrown? 3 heads, 2 heads and one tail, 2 tails and one head, three tails.

6:	$1 + 3 + 6$
6:	$1 + 4 + 5$
6:	$2 + 3 + 5$
3:	$2 + 2 + 6$
3:	$2 + 4 + 4$
3:	$3 + 3 + 4$
27	

SUGGESTION: Two heads and one tail may be regarded as a composite event of three constituent events as indicated at the right. The three coins are indicated by the letters *A*, *B*, *C*.

A	*B*	*C*
H	*H*	*T*
H	*T*	*H*
T	*H*	*H*

Compare the number of combinations of 3 heads and 3 tails, of 2 heads and one tail, and of one head and 2 tails.

The scheme at the right shows the number of combinations of 2 heads and one tail.

257. ***Mathematical expectation.***—If the "odds" on a certain event are m to n, this means that the probability of a favorable outcome is $m/(m+n)$. Thus if the odds on a certain game are 3 to 2 in favor of A, then the probability that A will win is taken as $3/(3+2) = 3/5$. If the winner of the game gets \$100, then A's mathematical expectation is $3/5 \times \$100 = \$60$.

It should be clear that the above is simply a definition of the meaning of the expression "the odds are m to n." If the "odds are even" then the probability is $1/(1+1) = \frac{1}{2}$.

Problem. A cash drawer contains an equal number of bills of the denominations \$1, \$5, \$10, \$20. What is the mathematical expectation of one who draws one bill at random from this drawer?

Solution: The probability that he will draw any one of these bills is ¼. Hence his expectation is ¼ × \$1. + ¼ × \$5. + ¼ × \$10. + ¼ × \$20.

¼ × \$1 =	\$0.25
¼ × \$5 =	1.25
¼ × \$10 =	2.50
¼ × \$20 =	5.00
	\$9.00

SIGHT WORK

1. The odds that a horse will win a certain race with purse of \$1000 are 5 to 4. What is the mathematical expectation on this race?

2. At a turkey shoot the probability that a man will kill the turkey at one shot is 1/6. If the turkey is worth \$3, what is the man's mathematical expectation if he has two shots?

3. On throwing a die once what is the probability that a six will not be thrown?

4. On throwing two dice what is the probability that two 3's will be thrown?

5. A player is to get \$4.50 if on throwing two dice he throws eleven (shows 11 points). What is his mathematical expectation on this throw?

6. A bag contains 5 red balls and 3 white balls. What is the probability that 2 red balls will be drawn in succession? What is the probability that 3 white balls will be drawn in succession?

7. If in problem 6 a ball is drawn and then put back into the bag, what is the probability that 2 red balls will be drawn in succession? What is the probability that 3 white balls will be drawn in succession? Compare the results in problems 6 and 7. What is the reason for the difference?

8. The odds that horse A will win a race are 2 to 7, while the odds on B winning are 2 to 11. What is the mathematical expectation of the man owning A and B if the stake is \$7500?

9. If three letters are taken at random from the word *Tennessee*, what is the probability that these letters will be e, n, and s?

258. Probability of mutually exclusive events.—An example will illustrate: On striking out one letter at random from the word *expectation*, what is the probability that either *n* or *t* will be stricken out?

SOLUTION: The probability that *n* will be stricken out is 1/11 and the probability that *t* will be stricken out is 2/11. Hence the probability that either of these will be stricken out is $1/11 + 2/11 = 3/11$. This is an instance of the general theorem:

The probability that some one of a set of mutually exclusive events will happen is the sum of the probabilities that the separate events will happen.

MISCELLANEOUS EXERCISES

1. From a pack of 52 cards two cards are drawn. (a) What is the probability that both will be spades?
   (b) What is the probability that they will be of the same color?
   (c) What is the probability that they will belong to the same suit?
   (d) What is the probability that one will be black and one red?
   (e) What is the probability that one will be a heart and one a spade?

2. From a bag containing 4 red balls and 7 black ones, two balls are drawn.
   What is the probability that
   (a) both will be red?
   (b) both will be black?
   (c) one will be red and one black?
   (d) at least one will be red?
   (e) at least one will be black?

3. If in exercise 2, a ball is drawn and placed back in the bag and then a second ball drawn, find the answers to the questions asked in that problem.

4. If 5 coins are thrown at one time, find the probability that at least one head will be thrown.
   What is the probability that exactly four tails will be thrown?
   Also find the probability that exactly one head will be thrown.

5. The probabilities that a horse will win each of three successive races are 1/3, 1/4, and 1/6 respectively. What is the probability that he will win at least one of them? That he will win some of them?

6. In a game of bridge, what is the probability that one player will get 13 cards of the same suit? What is the probability that he will get 13 spades?

7. From the words "probability" and "billboard," two letters are stricken out from each word. What is the probability (a) that two *b*'s will be stricken out from each word? (b) That at least one *b* will be stricken from each word? (c) That no *b* will be stricken from either word? (d) That *p* will be stricken from the first and *d* from the second?

259. Comparison of theoretical probability and results obtained in practice.—Probability as we have studied it thus far may be regarded as purely theoretical. We have assumed a numerical value of probability and then obtained our results from that assumption. Does our theory correspond to results obtained from experience? We shall see.

On throwing two dice the results given below were obtained. The dice were red and white respectively. We know that the theoretical probability of any pair such as *R*1, W1; *R*1, W2; *R*2, W1, and so on is 1/36. The following table shows the result of throwing two such dice 20,000 times.[1]

		White Die					
	No.	1	2	3	4	5	6
Red Die	1	547	587	500	462	621	690
	2	609	655	497	535	651	684
	3	514	540	468	438	587	629
	4	462	507	414	413	509	611
	5	551	562	499	506	658	672
	6	563	598	519	487	609	646

The combination *R*1, W1 occurred 547 times; *R*2, W1 occurred 609 times; *R*1, W2 occurred 587 times; and so on.

The theoretical probability for each combination is 1/36, and hence the number of times that each combination should occur is $1/36 \times 20{,}000 = 556$. The above results show considerable variation from this theoretical number. Is it likely that this may be due to the imperfection of the dice? Does 4 appear consistently less often than 6, for instance?

[1] These results were reported by R. Wolf, Bern, Switzerland. For reference see Czuber, *Wahrscheinlichkeitsrechnung*, p. 167.

Using 3 dice the author of this book obtained the results from 8640 throws that are shown below. A different set of dice was used for each set of 2160 throws and the results were added. The sums are given in the table.

This table shows that 3 points were thrown 40 times, 4 points were thrown 119 times, and so on.

Points	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Throws	40	119	257	417	576	808	990	1071	1057	993	862	592	387	268	154	49

EXERCISES

1. Find the theoretical probability for each of the numbers of points in the above table. How many throws should give each of the different numbers of points? Note that theoretically the number of 3's is the same as the number of 18's, that the number of 4's is the same as the number of 17's, and so on. Explain this symmetry in your results.

It has been suggested that a face having a comparatively large number of points shows oftener than one having a small number, because the points are made by gouging out a spot on the die and then putting a bit of paint in the hole, hence making the face lighter and thus more likely to turn up. Do the above results tend to confirm this idea? See also page 282.

2. Using two dice, compute the probability that 2 points will be thrown, that 3 points will be thrown, and so on up to 12 points. Using a total of 20,000 throws, compute to nearest unit the theoretical number of times that each number of points should be thrown.

3. From the table on page 282, compute the number of times that each of the number of points from 2 to 12 was actually thrown. The table gives at once that 2 points were thrown 547 times.

Three points were thrown $587 + 609 = 1196$ times.

Four points were thrown $500 + 655 + 514 = 1669$ times.

Five points were thrown $462 + 447 + 540 + 462 = 1961$ times, and so on.

4. Find the average of the numbers in each column and in each row in the table on page 282. What discrepancies do you observe?

5. From the table above find the number of percents by which the actual number of throws showing each number of points differed from the theoretical number. Use the theoretical number as the base. Thus 4 points were thrown 119 times while the theoretical number is 120. This difference, 1, is .83% of 120.

6. Make a graph showing the numbers in the table at the top of this page. Also make a graph of the numbers obtained in exercise 3. Use the number of points as abscissas and the number of throws as ordinates.

260. The American Experience Table of Mortality.—For a large body of problems involving probabilities, the number of events are obtained from statistical investigation. Prominent among the results of such investigations is the American Experience Table of Mortality compiled by our insurance companies. This table shows that of 100,000 white males living at the age of 10 years, 99,251 will be living one year later, 98,505 will be living two years later, and so on.

The actual method used in making this table was quite complicated. It was not possible to start with 100,000 all of age 10. For one thing it would be extremely difficult to get exact reports of the deaths of these particular individuals. Also, it would require 85 years to complete the investigation.

However, insurance companies could find by pooling this information that out of a large number insured at a certain age a certain number died each year. Hence the probability of dying, or living, was found, and thus this ideal table was finally constructed. It is the table now used by practically all our life insurance companies. It is known that in this table, for certain ages and under present conditions, the probability of living is understated and allowance for this is made in practical application. In fact, information on this point is constantly being accumulated.

Age	Number Living	Age	Number Living
10	100,000	53	66,797
11	99,251	54	65,706
12	98,505	55	64,563
13	97,762	56	63,364
14	97,022	57	62,104
15	96,285	58	60,779
16	95,550	59	59,385
17	94,818	60	57,917
18	94,089	61	56,371
19	93,362	62	54,743
20	92,637	63	53,030
21	91,914	64	51,230
22	91,192	65	49,341
23	90,471	66	47,361
24	89,751	67	45,291
25	89,032	68	43,133
26	88,314	69	40,890
27	87,596	70	38,569
28	86,878	71	36,178
29	86,160	72	33,730
30	85,441	73	31,243
31	84,721	74	28,738
32	84,000	75	26,237
33	83,277	76	23,761
34	82,551	77	21,330
35	81,822	78	18,961
36	81,090	79	16,670
37	80,353	80	14,474
38	79,611	81	12,383
39	78,862	82	10,419
40	78,106	83	8,603
41	77,341	84	6,955
42	76,567	85	5,485
43	75,782	86	4,193
44	74,985	87	3,079
45	74,173	88	2,146
46	73,345	89	1,402
47	72,497	90	847
48	71,627	91	462
49	70,731	92	216
50	69,804	93	79
51	68,842	94	21
52	67,841	95	3

PROBLEMS

In the problems on this page express probabilities in decimals correct to four decimal places.

1. What is the probability that a person, age 10, will live until he is 11 years old? What is the probability that he will die during his eleventh year?

2. What is the probability that a person, age 10, will live until he is 20 years old? 30 years old? 40 years old? 50 years old? 60 years old? 70 years old? 80 years old?

3. What is the probability that a person, age 20, will die in one year? SUGGESTION: How many of the 100,000 will be living at age 20? How many of these will die during the next year?

4. What is the probability that a person, age 10, will die in his 21st year? To solve this problem we must find (a) the probability that he will live to be 20 years old, (b) that in case he lives to be 20, he will die during the next year.

5. What is the probability that a person, age 20, will live to be 40 years old?

6. What is the probability that of two persons, age 20, both will die within 20 years?

7. What is the probability that two persons, each age 20, will both live to be 50 years old or more?

8. What is the probability that of the two persons in problem 7 one will die before he is 50 years old and one will live to be older than 50?

9. A man age 30 and his wife age 25 take out a life insurance policy containing a number of provisions. In figuring the cost of this policy it is necessary to know the probability that at the end of 20 years (a) both will be living, (b) both will be dead, (c) the man will be dead and the woman living, (d) the woman will be dead and the man living. Find each of these probabilities.

Use table page 284 for the wife also.

10. A man, age 40, has a legacy bequeathing him \$10,000 in case he lives to be 65. If he dies before that time the money is to go to charity. What is his mathematical expectation? (See page 282.) If the rate of interest is 3%, what is the present value of this expectation? (See page 217.)

If a man, age 40, were to buy such an expectation, would you regard that as a pure gamble? Is life insurance in general a gamble?

11. A husband and wife, each age 40, make a will bequeathing \$250,000 to a university to be paid 35 years from the date the will is made provided that at that time they are both dead. What is the probability that the university will receive their bequest? What is the mathematical expectation of this bequest? What is the present value of this expectation, if the rate of interest is 3%?

261. Proofs and theorems.—If there are n events equally likely to occur and if m of these are "favorable" (see §251), and consequently $n - m$ are unfavorable, and if one of these events is to occur, then the mathematical probability that this event will be favorable is m/n and that it will be unfavorable is $(n - m)/n$. In effect we define these numbers as the respective probabilities. The question as to whether these probabilities, as so defined, correspond to facts in the actual world of events has already been considered. (See §259.) We often refer to a favorable event as a success and an unfavorable event as a failure.

If there are a possible successes and b possible failures, then the total number of events is $a + b$, and the probability of success is $a/(a + b)$ and the probability of failure is $b/(a + b)$.

In a possible successes and b possible failures we have

Probability of success $\dfrac{a}{a+b}$

Probability of failure $\dfrac{b}{a+b}$

Theorem. If a composite event E *consists of* n *simple events* $e_1, e_2, \ldots, e_n$ *with a probability* p_1 *that* e_1 *is a success,* p_2 *that* e_2 *is a success, and so on, then the probability that the event* E *will be composed of* n *successful events is* $p_1 \cdot p_2 \cdot \ldots \cdot p_n$.

PROOF. Consider a composite event E that is a composite of two simple events e_1 and e_2. Suppose that event e_1 can occur in b ways, a of which are favorable, and that e_2 can occur in d ways, c of which are favorable. A combination of any one of the b events in e_1 with any one of the d events in e_2 constitutes an event E. Clearly there are bd such events. Again, a combination of any one of the a favorable events in e_1 with any one of the c favorable events in e_2 constitutes an event E. Hence there are ac such combinations, each one of which by definition is a success. Hence we have a total of bd events, ac of which are successes. Therefore the probability of a successful event E is ac/bd; but the probabilities of success in e_1 and e_2 are $a/b = p_1$ and $c/d = p_2$ respectively. Since $a/b \cdot c/d = ac/bd = p_1 \cdot p_2$, the theorem is proved for the case when the event E has only two constituent events. It is of course understood that success or failure of e_1 and e_2 are entirely independent of each other. Extension of the proof to the general case is now obvious.

Theorem. If an event can occur in any one of several mutually exclusive ways, then the sum of the probabilities that it will occur these different ways is unity. (See §253.)

PROOF. Let n be the total number of ways in which the event can occur. Separate the n ways in which the event can occur into two groups such that the first of these groups contains m of the ways in which the event can happen; the second group therefore contains $n - m$ of the ways in which it can happen. Hence the probabilities that the event will fall in each of these groups are m/n and $(n - m)/n$. Since the sum of these probabilities is unity, the theorem is proved for the case of two sets of mutually exclusive events. The extension to the general case is now obvious.

262. ***A general theorem on probability.***—As a preparation for the theorem on page 288 we shall study the following problems.

Problem 1. In throwing a die five times, what is the probability of throwing a 1 on each of the first 3 trials and failing to throw a 1 on the last two trials?

SOLUTION: The probability of throwing a 1 in each of the first three trials is 1/6, and hence the probability of throwing a 1 in all of these trials is $1/6 \cdot 1/6 \cdot 1/6 = (1/6)^3$. The probability of not throwing a 1 in each of the last two trials is 5/6 and hence the probability of not throwing a 1 in both is $(5/6)^2$. Therefore the required probability is $(1/6)^3 \cdot (5/6)^2 = 25/7776$.

Problem 2. In throwing a die five times, what is the probability of throwing a 1 in the first and second trials and failing to throw a 1 in the last three trials?

SOLUTION: The probability of throwing a 1 on each of the first two trials is 1/6, and $1/6 \cdot 1/6 = 1/36$ of throwing a 1 in both trials. The probability of not throwing 1 in one of the last three trials is 5/6, and $(5/6)^3$ is the probability of failing in all three trials. Hence the required probability is $(1/6)^2 \cdot (5/6)^3 = (1/6)^5 \cdot 5^3 = 125/7776$.

Clearly, the probability of success on the first and third trials and failure on the others is exactly the same as the probability of success in the first and second and failure in the others, and so for any two specified trials of the five.

EXERCISE

Find the probability of throwing a 1 in the first and third of 7 trials in throwing a die and failing to do so in the other 5 trials.

Problem 3. Find the probability of throwing a 1 exactly 3 times in 5 trials.

SOLUTION: By the method of Problem 1, page 287, we know that the probability of throwing 1 in each of any specified 3 trials and failing in the other two is $(1/6)^3 \cdot (5/6)^2$.

To solve the problem we must next find in how many ways the three trials may be taken. If we number first trial 1, the second 2, and so on, the individual combinations may be represented by 123, 124, 125, 134, 135, 145, 234, 235, 245, 345. That is, there are 10 such combinations. Evidently this is ${}_5C_3$ (see page 268). Since each of these combinations yields a probability of $(1/6)^3 \cdot (5/6)^2$, the answer is $10(1/6)^3 \cdot (5/6)^2 = 125/3888$.

Problem 4. In Problem 3, what is the probability of throwing a 1 at least 3 times?

SUGGESTION: Find the sum of the probabilities of throwing a 1 exactly five times, exactly four times, and exactly three times.

Theorem. If the probability that an event will occur in one trial is p *and the probability that it will fail is* q, *then the probability that it will occur exactly* r *times in* n *trials is* $\dfrac{n!}{r!(n-r)!} \cdot p^r q^{n-r}$.

PROOF. If any set of r events out of n events is selected, then the probability that all of these r events will be favorable is p^r and the probability that the remaining $n - r$ events will be unfavorable is q^{n-r}. Hence the probability that all of these r events will be favorable and the rest unfavorable is $p^r q^{n-r}$.

But a set of r events can be selected in ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ ways from n events. The probability then that in n trials the event will be favorable r times is $\dfrac{n!}{r!(n-r)!} p^r q^{n-r}$.

263. ***The binomial expansion of $(p + q)^n$.***—As above, let p be the probability that an event will be favorable and q the probability that it will be unfavorable. Then it turns out that the terms in the expansion of $(p + q)^n$ are the values of the expression $\dfrac{n!}{r!(n-r)!} p^r q^{n-r}$ for $r = n, n - 1$, and so on to $r = 0$. The kth term in the expansion

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2!}p^{n-2}q^2 + \ldots + q^n$$

is $\dfrac{n(n-1)\ldots(n-k+2)}{(k-1)!}p^{n-k+1}q^{k-1}$

$$= \frac{n!}{(n-k+1)!(k-1)!}p^{n-k+1}q^{k-1}.$$

If $n-k+1=r$, then $k-1=n-r$; hence this general term reduces to $\dfrac{n!}{r!(n-r)!}p^r q^{n-r} = {}_nC_r p^r q^{n-r}$. See pages 294, 295.

Thus, for example, in the expansion

$$(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5,$$

the first term is the probability that the event will be favorable 5 times in 5 trials (and hence unfavorable 5 times); the second term is the probability that the event will be favorable exactly 4 times in 5 trials (and hence unfavorable once); the third term is the probability that it will be favorable 3 times (and hence unfavorable twice); and so on.

The probability that the event will occur at least twice is the sum of the first four terms, since it will occur at least twice if it occurs 5, or 4, or 3, or 2 times. Since $p+q=1$, and hence $(p+q)^n=1$, this probability may be found by subtracting the sum of the last two terms from unity. The probability that the event will occur less than four times is the sum of the last two terms, since these are the probabilities that it will fail 5 times and 4 times.

Problem. The probability that a man will hit a target is 4/5. What is the probability (a) that he will hit it exactly 3 times in 5 trials? (b) that he will hit it at least 3 times in 5 trials? (c) that he will hit it less than 3 times in 5 trials?

SOLUTION: The probability that he will not hit it in one trial is 1/5. Hence in our formula $p = 4/5$ and $q = 1/5$.

$$\left(\frac{4}{5}+\frac{1}{5}\right)^5 = (4/5)^5 + 5(4/5)^4 \cdot 1/5 + 10(4/5)^3(1/5)^2 + 10(4/5)^2(1/5)^3 + 5(4/5)(1/5)^4 + (1/5)^5.$$

By the above the answer to (a) is $10(4/5)^3(1/5)^2 = 128/625$.

The answer to (b) is $(4/5)^5 + 5(4/5)^4(1/5) + 10(4/5)^3(1/5)^2 = 2944/3125$.

The answer to (c) is $(1/5)^5 + 5(4/5)(1/5)^4 + 10(4/5)^2 \cdot (1/5)^3 = 180/3125$.

Check the computation in the above.

PROBLEMS

1. Six books marked A, B, C, D, E, F are placed at random side by side on a shelf. What is the probability that A stands at the left end and F at the right end?

2. In problem 1 what is the probability that on one trial the books will all be in any one specified order, such as $ABCDEF$?

3. If one picks at random three of these 6 books, what is the probability that books A, B, C will be taken?

4. On 4 trials what is the probability that one will pick books A, B, C exactly twice?

5. In the same problem what is the probability that books A, B, C will be picked at least three times? What is the probability that they will be picked less than 4 times?

6. If five letters A, B, C, D, E are placed at random in a line, what is the probability that A stands immediately before B?

7. In a bag there are 3 white balls and 4 black balls. A man takes out a ball several times and then replaces it in the bag. What is the probability that on 5 trials he will take 5 white balls? 5 black balls?

8. If in the preceding problem a man takes out a ball four times, what is the probability that he will take exactly 1 white ball? 2 white balls? What is the probability that he will take out exactly 1 black ball? 2 black balls? 3 black balls? 4 black balls?

9. In the preceding problem, what is the probability that he will take out at least 3 white balls on 4 trials? that he will take out less than 3 white balls on 4 trials?

10. Five letters are written and five envelopes addressed. If the letters are placed in the envelopes at random, what is the probability that all the letters will be placed correctly? that they are all placed incorrectly? that exactly 3 are placed correctly?

11. A coin is tossed 7 times. What is the probability of 5 heads and 2 tails?

12. A bag contains 2 black balls, 4 red balls, and 6 white balls. Two balls are drawn. What is the probability of drawing exactly 2 black balls? 2 red balls? 2 white balls? (a) if the first ball is not replaced before drawing the second? (b) if the first ball is put back in the bag before drawing the second?

CHAPTER 22:

THE BINOMIAL THEOREM; MATHEMATICAL INDUCTION

The first purpose of this chapter is to develop what is called the binomial theorem by means of which $(a + b)^n$ may be written as a polynomial for any positive integral value of n. The second purpose is to study a method of proof called mathematical induction. Using this method we shall make a second proof of the binomial theorem.

264. *Exponents in the binomial expansion.*—By multiplication we can verify the following.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

In each of these expansions the exponent of a in the first term is the same as the exponent of the binomial. The exponent of a then decreases by 1 in each succeeding term, until in the last term it becomes zero, giving $a^0 = 1$, and a thus disappears. In the first term b does not occur, or we may regard it as appearing with an exponent zero. The exponent of b in the second term is 1 and then increases by 1 in each succeeding term until in the last term it reaches the exponent of the binomial.

This holds for any positive integral value of the exponent of the binomial. That is, in the expansion of $(a + b)^n$ for any such value of n, the first term is a^n. The exponents of a in succeeding terms decrease as above. The exponent of b in the second term is 1 and then increases until it reaches n in the last term, which is b^n. That is, omitting the coefficients, the terms of the expansion of $(a + b)^n$ are

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ \ldots,\ a^2b^{n-2},\ ab^{n-1},\ b^n.$$

It follows that there are $n + 1$ terms in the expansion of $(a + b)^n$.

265. The coefficients in the binomial expansion.—Consider the expansion of $(a+b)^5$ given on page 291.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

We are now seeking a rule by which the coefficients of the expansion of $(a+b)^n$ can be written without performing the actual multiplication. The following is an old rule for doing this.

1. The coefficient of the first term is 1.

2. The coefficient of the second term is the same as the exponent of a in the first term.

3. To obtain the coefficient of any term after the second, multiply the coefficient in the preceding term by the exponent of a in that term and divide by the exponent of b in that term increased by 1.

Using this rule, we may write the above expansion.

$$a^5 + 5a^4b + \frac{5 \cdot 4}{2}a^3b^2 + \frac{5 \cdot 4 \cdot 3}{2 \cdot 3}a^2b^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 4 \cdot 5}b^5$$

Note that writing a number in the numerator as shown here multiplies the term by that number, while writing it in the denominator divides it by that number. When the fractions are reduced this is identical with the above expansion. This rule together with the rule for writing the exponents given on page 291 gives the expansion of $(a+b)^n$ as follows.

$$a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^{n-4}b^4 + \dots$$

This is the general expansion that we are to study.

EXERCISES

1. By multiplication verify that the expansion of $(a-b)^5$ is the same as that of $(a+b)^5$ with the exception that in the former alternate terms are negative. This holds for the general expansion of $(a-b)^n$.
2. Using the above formula, write the expansion of $(a+b)^6$ and $(a-b)^6$.
3. Write the expansions of $(x+1)^7$, $(x-1)^7$, $(1+x)^7$, $(1-x)^7$.
4. Write the 6th term in $(a+b)^{10}$ without writing the preceding terms.

266. The Pascal triangle.—The triangular array of numbers shown below, called the Pascal triangle, shows the coefficients of the expansions of $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$, . . . In this triangle the first and the last figure in each line is unity. Every other figure is the sum of the figure directly above it and the one directly to the left of that.

$(a+b)^0$	1										
$(a+b)^1$	1	1									
$(a+b)^2$	1	2	1								
$(a+b)^3$	1	3	3	1							
$(a+b)^4$	1	4	6	4	1						
$(a+b)^5$	1	5	10	10	5	1					
$(a+b)^6$	1	6	15	20	15	6	1				
$(a+b)^7$	1	7	21	35	35	21	7	1			
$(a+b)^8$	1	8	28	56	70	56	28	8	1		
$(a+b)^9$	1	9	36	84	126	126	84	36	9	1	
$(a+b)^{10}$	1	10	45	120	210	252	210	120	45	10	1
- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -

Thus, in the line for $(a+b)^6$, $1+5=6$, $5+10=15$, $10+10=20$, $10+5=15$, $5+1=6$.

Using this scheme, a table of coefficients may be extended as far as we wish. However, the scheme cannot be used directly to expand, for example, $(a+b)^{25}$, whereas the formula given on page 292 may be used for this purpose. If in the expansion of $(a+b)^6$, $a=1$ and $b=1$, then each term will equal its coefficient $(a^kb^h = 1^k \cdot 1^h = 1)$. Hence the sum of the coefficients must equal $(1+1)^6 = 2^6$. In the expansion $(a-b)^n$, the sum of the coefficients is $(1-1)^n = 0$.

EXERCISES

1. In the expansion of $(a+b)^4$, what is the sum of the coefficients?

2. What is the sum of the coefficients in the expansion of $(a+b)^5$? in $(a+b)^7$? in $(a+b)^{10}$? Note that each of these involves a term of a geometric series.

3. What is the sum of all the coefficients shown in the above Pascal triangle? What is this sum if the triangle is continued to show the expansion of $(a+b)^n$? Note that the required sum is the sum of a geometric series.

4. What is the sum of the coefficients in the expansion of $(a-b)^4$? of $(a-b)^7$? of $(a-b)^{10}$? of $(a-b)^n$?

267. Discovery and proof of the binomial theorem.—To discover and prove the binomial theorem we shall study the product

$$(x + a_1)(x + a_2)(x + a_3)(x + a_4) \dots (x + a_n) \qquad (1)$$

If we use only three of these factors and expand the products we have

$$(x + a_1)(x + a_2)(x + a_3) = x^3 + (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x + a_1a_2a_3 \qquad (2)$$

We may think of obtaining the product in this way:

Every term in it is the product of three factors, one from each parenthesis. If we take x from each of the three parentheses, we have x^3. If we take x from two parentheses and a_1, a_2, a_3 in succession from the others, we have a_1x^2, a_2x^2, a_3x^2, or $(a_1 + a_2 + a_3)x^2$. If we take x from one parenthesis and two of the letters a_1, a_2, a_3 from the others, we have $a_1a_2x + a_1a_3x + a_2a_3x$, or $(a_1a_2 + a_1a_3 + a_2a_3)x$. Finally, if we take a_1, a_2, a_3 from the three parentheses we have $a_1a_2a_3$.

If we put $a_2 = a_1$, $a_3 = a_1$, (2) reduces to

$$(x + a_1)^3 = x^3 + 3a_1x^2 + 3a_1^2x + a_1^3.$$

If we find the product in (1) in this way, we shall have x^n as the first term, $(a_1 + a_3 + a_3 + \dots + a_n)x^{n-1}$ as the second term, $(a_1a_2 + a_1a_3 + \dots)x^{n-2}$ as the third term. If we let $S_1, S_2, S_3 \dots$ be the number of terms in these parentheses and let $a_1 = a_2 = a_3 \dots$, then the product becomes

$$x^n + S_1x^{n-1}a_1 + S_2x^{n-2}a_1^2 + S_3x^{n-3}a_1^3 + \dots$$

But what are the values of $S_1, S_2, S_3, \dots$?

We have n a's in $(x + a_1)(x + a_2) \dots$ Hence S_1 is the number of combinations in n objects taken one at a time, or ${}_nC_1$. S_2 is the number of pairs of objects in the n constants, or ${}_nC_2$. Similarly, $S_3 = {}_nC_3$, $S_4 = {}_nC_4$, and so on.

Hence we have

$$\begin{aligned}(x + a)^n &= x^n + S_1x^{n-1}a + S_2x^{n-2}a^2 + S_3x^{n-3}a^3 + \dots \\ &= x^n + {}_nC_1x^{n-1}a + {}_nC_2x^{n-2}a^2 + {}_nC_3x^{n-3}a^3 + \dots \\ &= x^n + nx^{n-1}a + \frac{n(n-1)}{1 \cdot 2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}a^3 \\ &\quad + \dots,\end{aligned}$$

which is the binomial theorem.

In this formula the sequence of numbers

$$1,\ n,\ \frac{n(n-1)}{1\cdot 2},\ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3},\ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4},\ \ldots$$

are called the binomial coefficients. For any value of n (integral and positive) there are $n+1$ terms in this sequence. If we change the sign of a, we have $(x-a)^n$. In this case the signs in the product are alternately positive and negative, so that the sequence is

$$1,\ -n,\ \frac{n(n-1)}{1\cdot 2},\ -\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3},\ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4},\ \ldots$$

When these coefficients are written in the form

$$1,\ {}_nC_1,\ {}_nC_2,\ {}_nC_3,\ \ldots\ {}_nC_{n-2},\ {}_nC_{n-1},\ {}_nC_n,$$

we recall (see page 268) that ${}_nC_r = {}_nC_{n-r}$, and we see that

$$1 = {}_nC_n,\ {}_nC_1 = {}_nC_{n-1},\ {}_nC_2 = {}_nC_{n-2},\ \ldots$$

That is, the first and the last terms are numerically equal (both equal to 1), the second term and the second from the last end are numerically equal, and so on. That is, in $(a+b)^{47}$ the third term, $\frac{n(n-1)}{1\cdot 2}a^{45}b^2$, and the 46th term, the third from the last, have the same coefficient. Hence to find the 46th term in this expansion, we may take the coefficient of the third term.

That is, the third term of this expansion is $\frac{47\cdot 46}{2}a^{45}b^2 = 1081\ a^{45}b^2$, and the 46th term is $1081\ a^2b^{45}$.

We also note the following.

(a) If the power of the expansion is even, there is an odd number of terms and the coefficients are symmetrical with respect to the middle term.

(b) If the power of the expansion is odd, there is an even number of terms and there are two equal terms in the middle. The remaining coefficients are symmetrical with respect to these.

Thus, in $(a+b)^6$ the coefficients are 1, 6, 15, 20, 15, 6, 1, in which 20 is the middle one and the others are symmetrical with respect to it.

In $(a+b)^7$ the coefficients are 1, 7, 21, 35, 35, 21, 7, 1, in which 35, 35 are in the middle and the remaining terms are symmetrical with respect to these.

EXERCISES

1. Find the sum of the coefficients in the expansion of $(x+y)^{16}$.
2. Find the sum of the coefficients in the expansion of $(ax+y)^{20}$.
3. Expand $(\frac{1}{2}x+3y^2)^4$ and simplify.
4. Find the middle term in the expansion of $(x^2-y)^8$ and simplify.
5. Find the middle terms in the expansion of $(a^{1/3}+y)^7$.
6. Write the expansion of $(1+1)^7$ to show that the sum of the coefficients is 2^7.
7. Using Pascal's triangle write the coefficients in the expansion $(a+b)^{11}$, $(a+b)^{12}$, $(a+b)^{13}$. Start with the triangle on page 293.
8. Find, in simplified form, the middle term of $(a^{2/3}-2a^{-1/2})^8$.
9. Find the 35th term in the expansion of $(a+b)^{37}$.

268. *The binomial theorem with fractional exponents.*—The coefficient in the general, or rth, term in the binomial expansion is

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}.$$

When $r=n+2$, the last factor in the numerator is zero, and hence the product is zero. This factor zero occurs in all subsequent terms of the expansion and hence all terms after the $(n+1)$st term are zero and the expansion terminates. If the exponent n is a fraction no factor in a numerator of the coefficients in the expansion is zero and the terms continue indefinitely. That is, we have an infinite series. (See Chapter 23.) No proofs for this case will be attempted in this book, but we shall state the following theorem.

If a *is numerically greater than* b *and the exponent* k *is a fraction, then the binomial expansion may be used to approximate the value of* $(a+b)^k$.

Thus, $\sqrt{10}=\sqrt{9+1}=(9+1)^{1/2}=3\left(1+\frac{1}{9}\right)^{1/2}$

$$=3\left[1+\frac{1}{2}\cdot\frac{1}{9}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{9}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2\cdot 3}\left(\frac{1}{9}\right)^3+\ldots\right]$$

$$=3(1+0.05556-0.00154+0.00009+\ldots)=3.16233$$

Evidently the terms in this expansion are growing small very rapidly, and taking additional terms would not change the result in the fourth decimal place.

Taking successive terms of this series serves to approximate the value of $\sqrt{10}$ in much the same way that extending the number of decimals in $1/3 = 0.333\ldots$ approximates the value of $1/3$. If we try to expand $(1+9)^{1/2}$ instead of $(9+1)^{1/2}$ we shall not get an approximation to $\sqrt{10}$ as the number of terms in the expansion is increased.

In approximating $\sqrt{a}$ by this method we separate a into two numbers, the larger of which is a square. Thus for $\sqrt{5}$ we write $(4+1)^{1/2} = 2\left(1+\frac{1}{4}\right)^{1/2}$, for

$$\sqrt{3} = (4-1)^{1/2} = 2\left(1-\frac{1}{4}\right)^{1/2},$$

and so on. At the right we have given a few further examples.

$$\sqrt{7} = (9-2)^{1/2} = 3\left(1-\frac{2}{9}\right)^{1/2}$$
$$\sqrt{15} = (16-1)^{1/2} = 4\left(1-\frac{1}{16}\right)^{1/2}$$
$$\sqrt{57} = (49+8)^{1/2} = 7\left(1+\frac{8}{49}\right)^{1/2}$$

Instead of $\sqrt{57} = (49+8)^{1/2}$ we may use $\sqrt{57} = (64-7)^{1/2}$. There are three points to be observed:

a. The first number must be a square.

b. The second number must be numerically less than the first.

c. The second number should be numerically as small as possible.

The binomial expansion may also be used when the exponent is negative, but we shall not continue this subject further. A fuller treatment is given in texts on calculus.

EXERCISES

1. Using the method of §268, find $\sqrt{3}$ correct to four places of decimals.
2. Using the same method, approximate $\sqrt{29}$, $\sqrt{34}$, $\sqrt{94}$, $\sqrt{987}$.
3. Write the first four terms in the expansions of $\left(27-\frac{2}{a}\right)^{1/3}$ and $(16-3)^{3/4}$.
4. Write, in simplest form, the seventh term of the expansion of $(1-x)^{1/2}$. For what values of x may this expansion be used?
5. Find, in the simplified form, the term involving x^3 in the expansion of $(a-x^{1/2})^{10}$.
6. Write the middle term of the expansion of $(a^{-1/2}-a^{3/4})^8$ and reduce it to the simplest form.

269. Mathematical induction.—The general scheme in mathematical induction is illustrated in the following.

Suppose we wish to prove that $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$ for all positive integral values of n.

STEP 1. Suppose that this formula holds for some value of n as $n = k$.

Then,
$$1 + 2 + 3 + \ldots + k = \frac{k}{2}(k + 1).$$

Adding $k + 1$, the next term in the series, to both members,

$$1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k}{2}(k + 1) + k + 1.$$

But, $\frac{k}{2}(k + 1) + k + 1 = (k + 1)\left(\frac{k}{2} + 1\right) = \frac{1}{2}(k + 1)(k + 2).$

And hence, $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{1}{2}(k + 1)(k + 2)$, which is the value of $\frac{n}{2}(n + 1)$ when n is replaced by $k + 1$.

Hence, *if* the formula holds for $n = k$, then it holds for $n = k + 1$.

STEP 2. But we know that the formula holds for $n = 1$. Therefore it holds for $n = 2$, and hence it holds for $n = 3$, and so on as far as we care to go. That is, the formula holds for any positive integral value of n.

These two steps in the proof are further illustrated in the examples that follow.

Example 1. Prove that $x - y$ is a factor of $x^n - y^n$ for any positive integral value whatever of n.

PROOF: Suppose that $x - y$ is a factor of $x^k - y^k$ for a certain positive integer k.

Since
$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - xy^k + xy^k - y^{k+1} \\ &= x(x^k - y^k) + y^k(x - y), \end{aligned}$$
it follows that $x - y$ is a factor of $x^{k+1} - y^{k+1}$.

That is, if for any value of the exponent k, $x - y$ is a factor of $x^k - y^k$, then $x - y$ is a factor of this expression when the exponents are increased by unity.

But we know that $x - y$ is a factor of $x^2 - y^2$. Hence it follows by the above argument that $x - y$ is a factor when the exponents are increased by unity. That is, $x - y$ is a factor of $x^3 - y^3$. By exactly the same argument it now follows that $x - y$ is a factor of $x^4 - y^4$. By successive repetitions of this argument it follows that $x - y$ is a factor of $x^5 - y^5$, of $x^6 - y^6$, and so on for any positive integral exponent whatever.

The essential steps in this argument are:

(a) *From the hypothesis that a proposition holds for a certain value of a letter* (k *for instance*) *it is proved that the proposition holds when the value of this letter is increased by unity.*

(b) *It is proved that the proposition holds for a certain positive integral value, as* 1 *or* 2, *of this letter.*

It then follows that the proposition holds successively for all integral values of the letter which are greater than 1 or 2.

Example 2. To illustrate the necessity of both (a) and (b) in this argument we will attempt to prove that $x - y$ is a factor of $x^n + y^n$.

Since $x^{k+1} + y^{k+1} = x(x^k + y^k) - (x - y)y^k$, it follows that if $x - y$ is a factor of $x^k + y^k$, then it is a factor of $x^{k+1} + y^{k+1}$.

That is, step (a) of the argument has been carried through.

When we attempt to carry through step (b) of the argument, we are unable to find a value of n for which $x - y$ is a factor of $x^n + y^n$.

However, this argument does not prove that there are no integral values of n for which $x - y$ is a factor of $x^n + y^n$. Thus $x - y$ might conceivably be a factor of $x^{100} + y^{100}$. But we do know, however, that *if* $x - y$ is a factor of $x^{100} + y^{100}$ then it is a factor of $x^n + y^n$ for all integral values of n above 100.

That $x - y$ is not a factor of $x^n + y^n$ for any value of n is easily shown.

From the identity $x^{k+1} + y^{k+1} = x(x^k + y^k) - (x - y)y^k$ it follows that if $x - y$ is a factor of $x^{k+1} + y^{k+1}$, then it is a factor of $x^k + y^k$. Hence it follows that if $x - y$ is a factor of $x^{100} + y^{100}$ it is also a factor of $x^{99} + y^{99}$, and so on down to $x^2 + y^2$. But we know that $x - y$ is not a factor of $x^2 + y^2$ and hence it follows that it is not a factor of $x^n + y^n$ for any integral value of the exponent.

Example 3. Since we know that $x + y$ is a factor of $x^3 + y^3$ we may attempt to prove it is a factor of $x^n + y^n$ for all integral values of n greater than 3.

When we attempt to carry out step (a) of the proof we find

$$x^{k+1} + y^{k+1} = x(x^k + y^k) - (x - y)y^k.$$

From this it does not follow that if $x + y$ is a factor of $x^k + y^k$, then it is a factor of $x^{k+1} + y^{k+1}$. That is, we are unable to carry out this step of the argument.

The fact that we find ourselves unable to carry out this step is, of course, not a proof that it cannot be done. We know, however, for other reasons that this is the case. As a matter of fact $x + y$ is a factor of $x^n + y^n$ when n is an odd number and not when n is even.

Example 4. Prove that for all positive integral values of n

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Assuming that the formula holds for a certain positive integral value k of n we have,

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

Adding $(k+1)^3$ to both members of this equation gives

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3.$$

The second member of this equality may be written,

$$(k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\frac{(k^2+4k+4)}{4} = (k+1)^2\left(\frac{k+2}{2}\right)^2.$$

Hence we have an expression in which the k of $\left(\frac{k(k+1)}{2}\right)^2$ has been replaced by $k+1$. That is, step (a) of the argument has been carried through.

We now note that $1^3 + 2^3 = \left(\frac{2(2+1)}{2}\right)^2$, since both members of this equality are equal to 9.

Hence we have proved that the formula holds for all positive integral values of n.

From these examples it is apparent that proof by mathematical induction is applicable only when a formula is stated in terms of a letter n which can take the successive values 1, 2, 3, . . .

Example 5. Prove the binomial theorem by mathematical induction.

Assume that the formula (see page 294) holds for a certain value k of n. We have,

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 + \ldots \quad (1)$$

The next step is to multiply both members of this equality by $a + b$. We first multiply by a and then by b and arrange the terms as shown below.

$$(a+b)^{k+1} = a^{k+1} + ka^kb + \frac{k(k-1)}{2}a^{k-1}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-2}b^3$$

$$+ \quad a^kb + \qquad ka^{k-1}b^2 + \qquad \frac{k(k-1)}{2}a^{k-2}b^3 + \cdots$$

$$= a^{k+1} + (k+1)a^kb + \left[\frac{k(k-1)}{2} + k\right]a^{k-1}b^2 + \cdots$$

$$= a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{2\cdot 3}a^{k-2}b^3 \quad (2)$$

Note that $\dfrac{k(k-1)}{2} + k = k\left(\dfrac{k-1}{2} + 1\right) = k\left(\dfrac{k+1}{2}\right)$.

$$\frac{k(k-1)(k-2)}{2\cdot 3} + \frac{k(k-1)}{2} = \frac{k(k-1)}{2}\left[\frac{k-2}{3} + 1\right] = \frac{k(k-1)(k+1)}{2\cdot 3}.$$

The product is finally arranged as in (2). Comparing (1) and (2), we see that if in (1) we replace k by $k+1$ we have (2). Hence (1) and (2) are exactly the same formula. That is, if (1) holds for a value k of the exponent, then it holds for $k+1$. But we know (see page 291) that this formula holds for $k = 1$, or 2, or 3, . . . Hence it holds for any positive integral value of k.

EXERCISES

1. Using the expansion $(a+b)^3 = a^3 + 3a^2b + \frac{3(3-1)}{2}ab^2 + \frac{3(3-1)(3-2)}{2\cdot 3}b^3$ multiply both members by $a+b$ as above and verify that this formula holds for $(a+b)^4$.

2. Carry out completely the steps in Example 5 when $k = 5$.

3. Suppose that for a certain formula involving n, step (1) of the proof by induction has been carried out and also that the formula is found to hold for $n = 10$; what conclusion can be drawn?

4. Suppose it is proved that if a certain formula holds for $n = k$ then it holds for $n = k - 1$, and also that it does hold for $n = 10$; what conclusion can be drawn?

5. Write a full statement of the plan of proof used in mathematical induction.

EXERCISES

1. Prove by mathematical induction that the sum of $a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d]$ is $\frac{n}{2}[2a + (n - 1)d]$, or $\frac{n}{2}(a + l)$.

2. Prove the formula giving the sum of the geometric series $a + ar + ar^2 + \ldots + ar^{n-1}$.

Prove the following formulas.

3. $2 + 4 + 6 + \ldots + 2n = n(n + 1)$.
4. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
5. $3 + 6 + 9 + \ldots + 3n = \frac{3}{2}[n(n + 1)]$.
6. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$.
7. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n + 1)(n + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$.
8. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.
9. $2^2 + 4^2 + 6^2 + \ldots (2n)^2 = \frac{2}{3}n(n + 1)(2n + 1)$.

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.
11. $2 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$.
12. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}$.
13. $1 \cdot 1 + 2 \cdot 3^2 + 3 \cdot 5^2 + \ldots + n(2n - 1)^2 = \frac{1}{6}n(n + 1)(6n^2 - 2n - 1)$
14. $1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + \ldots + (2n - 1)3^n = 3^{n+1}(n - 1) + 3$.
15. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n}{2}(n + 1)\right]^2$.
16. $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \ldots + (n + 1)(n + 4) = \frac{n}{3}(n + 4)(n + 5)$.
17. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \ldots + 2n(2n + 2) = \frac{n}{3}(2n + 2)(2n + 4)$.
18. $1^3 + 2^3 + \ldots + n^3 - 1^2 - 2^2 - \ldots - n^2 = \frac{n}{12}(n^2 - 1)(3n + 2)$.
19. Prove that $x^n - y^n$ is divisible by $x - y$ by using

$$\frac{x^n - y^n}{x - y} = x^{n-1} + \frac{y(x^{n-1} - y^{n-1})}{x - y}.$$

CHAPTER 23:

INFINITE SERIES

When a series such as $1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$ is regarded as extending indefinitely, it is clear that all the terms cannot be added in the ordinary sense since the operation can never be completed. Hence, if we are to regard such a series as having an exact "sum," we must attach to this word a meaning in some respect different from that used in ordinary arithmetic and in elementary algebra. We shall now study a meaning of "sum" that will enable us to apply it to an endless (infinite) series.[1]

270. An example of the "sum" of an infinite series.—Let us consider the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}$$

when n is allowed to increase indefinitely. The sum of the first two terms is $1\frac{1}{2}$, the sum of the first three terms is $1\frac{3}{4}$, the sum of the first four is $1\frac{7}{8}$, and the sum of the first $n+1$ terms is $1+\frac{2^n-1}{2^n}=2-\frac{1}{2^n}$.

Since $\frac{1}{2^n}$ can be made as small as we wish by making n sufficiently large, it follows that the sum can be made to differ from 2 by as little as we please by taking a sufficient number of terms. That is, the sum "goes toward 2" as n is made larger and larger.

We make our statement precise as follows.

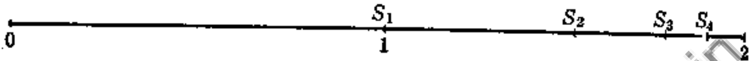
1. No matter how many terms of this series are added, the sum will be less than 2.

2. If k is any fixed number less than 2, then we can add enough terms of the series to make the sum greater than k.

[1] The word "infinite" has a negative meaning; it simply indicates "without end" or boundary. *Finis* in Latin means end or boundary.

271. Geometric representation of the sum of an infinite series.—In the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$, let S_n represent the sum of the first n terms.

Then, $S_1 = 1$, $S_2 = 1\frac{1}{2}$, $S_3 = 1\frac{3}{4}$, $S_4 = 1\frac{7}{8}$, . . .



As the number of terms increases, it is evident that the points representing the sums move toward the right but will never reach the point 2. The first step, from 0 to 1, takes half the distance from 0 to 2, the distance remaining to 2 being 1. The next step, from 1 to $1\frac{1}{2}$, takes half this remaining distance reaching $1\frac{1}{2}$, the remaining distance now being $\frac{1}{2}$. In general, each step takes half the remaining distance. By this process the whole distance is never used up but the remaining distance becomes as near zero as we please. So we say that as n becomes infinite (unbounded) the sum, S_n, approaches 2 as a limit. This is indicated by the expression at the right.

$$\lim_{n \to \infty} S_n = 2$$

272. A fundamental proposition on limits.—*If we have a sequence* S_n *of numbers such that each number is equal to, or greater than, the one that precedes it, and also such that there is some number greater than all numbers in the sequence, then there is a definite number* S *which this sequence approaches as a limit.*

$$\lim_{n \to \infty} S_n = S$$

The above sequence $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$ is of this kind. In this case each number is actually *greater* than the one that precedes it. But the situation would not be changed if we were to insert certain terms making the sequence read $1, 1\frac{1}{2}, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{7}{8}, 1\frac{7}{8}, 1\frac{15}{16}, \ldots$

In this sequence there are some terms each of which is *equal* to the one that precedes. But the limit approached remains unchanged. In this same sequence, the number 2, or any number greater than 2, is greater than every number of the sequence.

273. *An alternating series.*—Consider now the infinite series

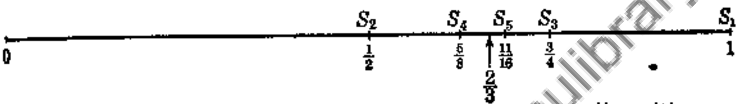
$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} \cdots$$

in which the first term is positive, the second negative, the third positive, and so on. Then the successive sums are

$$S_1 = 1,\ S_2 = \frac{1}{2},\ S_3 = \frac{3}{4},\ S_4 = \frac{5}{8},\ S_5 = \frac{11}{16}, \cdots$$

It is easily shown that as n grows large the sum approaches $\frac{2}{3}$ as a limit. (See page 306.)

The behavior of the sums is shown in the following.



The steps now go alternately forward and back. The steps decrease in length, but each one goes across the point $\frac{2}{3}$. The points to the left of $\frac{2}{3}$, $S_2, S_4, S_6, \ldots$ go toward $\frac{2}{3}$ exactly as the points on page 304 go toward 2. The points $S_1, S_3, S_5, S_7 \ldots$ also go toward the same point $\frac{2}{3}$. The distance from these points to $\frac{2}{3}$ decreases toward zero as we go on. In this way the sequence of sums approaches more and more nearly to the limiting point $\frac{2}{3}$, and this point is the limit of the points $S_1, S_2, S_3, \ldots S_n$ as n becomes infinite.

$$\lim_{n \to \infty} S_n = \frac{2}{3}$$

The fundamental proposition, page 304, applies directly to the increasing sequence S_2, S_4, S_6, and, with an obvious modification, to the decreasing sequence $S_1, S_3, S_5, S_7, \ldots$ In both cases the limit approached is the point $\frac{2}{3}$.

EXERCISES

1. Construct a figure representing the sequence of sums in the series $1 + \frac{3}{4} + \frac{9}{16} + \left(\frac{3}{4}\right)^3 + \ldots + \left(\frac{3}{4}\right)^n + \ldots.$ The limit of these sums is 4.

2. Construct a figure representing the sequence of sums in the series $1 - \frac{3}{4} + \frac{9}{16} - \left(\frac{3}{4}\right)^3 + \ldots.$ The limit of these sums is 4/7.

274. The sum of an infinite geometric series.—The general infinite geometric series (compare page 201) is

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots$$

where the series is regarded as continuing indefinitely. We say that the series has an infinite (unlimited) number of terms.

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\lim_{n \to \infty} S_n = \frac{a}{1-r}$$

From the formula on page 202 we know that the sum of the first n terms of this series is $\frac{a(1 - r^n)}{1 - r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$. The symbol S_n is used to designate this sum, the subscript n indicating the number of terms. The values of S_n for different values of n are called partial sums of the series. If r is between -1 and $+1$, r^n grows small in absolute value as n grows large. If n increases indefinitely, r^n takes values as near zero as we please. This means that $\frac{ar^n}{1-r}$ takes values as near zero as we please. Hence by taking n sufficiently large, the partial sums of the series can be made to take values as near $\frac{a}{1-r}$ as we please. This statement is written briefly $\lim_{n \to \infty} S_n = \frac{a}{1-r}$, which is read: "Limit S sub n as n becomes infinite is $\frac{a}{1-r}$."

If a and r are positive, all the terms of the series are positive and the sum approaches $\frac{a}{1-r}$ from below, or from the left. That is, the sum is always less than $\frac{a}{1-r}$. For every positive number d, however small, there is a value of n such that the sum of n terms of the series differs from $\frac{a}{1-r}$ by less than d for that value of n and for all greater values. This is the case described in §271.

If r is negative, then $\frac{ar^n}{1-r}$ is alternately negative and positive.

Hence the sum $\frac{a}{1-r} - \frac{ar^n}{1-r}$ is alternately greater and less than $\frac{a}{1-r}$, but it grows as "near" to $\frac{a}{1-r}$ as we please.

That is, for every positive number d, however small, there is a value n such that the sum of n terms differs from $\frac{a}{1-r}$ by less than d for that value of n and for all greater values. This is the case described in §273. All this, whether r is positive or negative, is indicated by $\lim_{n\to\infty} S_n = \frac{a}{1-r}$. This discussion is based on the assumption that r lies between -1 and $+1$, for it is only in that case that r^n grows small as n increases. We say that for these values of r the series is convergent.

If $r = 1$, the series is $a + a + a + \ldots$, the sum of which increases beyond all bounds as n grows large since $S_n = na$. This also holds for any value of r that is greater than 1. If $r = -1$, then the series is $a - a + a - a + a \ldots$, and the sum is alternately a and 0, and hence the sum does not approach any single limit as n grows large. If r is less than -1, -2 for instance, the terms grow large numerically and the sums are alternately positive and negative but differ increasingly as n grows large.

A series whose partial sum does not approach a definite finite limit as n grows large is said to be divergent. As we have seen, the geometric series is convergent for r between -1 and $+1$ and divergent for all other values of r.

EXERCISES

In each of these geometric series find the ratio and the sum to infinity.

1. $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \ldots$
2. $1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \ldots$
3. $1 + \frac{7}{8} + \left(\frac{7}{8}\right)^2 + \left(\frac{7}{8}\right)^3 + \ldots$
4. $1 - \frac{63}{64} + \left(\frac{63}{64}\right)^2 - \left(\frac{63}{64}\right)^3 + \ldots$
5. $1 + .999 + (.999)^2 + (.999)^3 + \ldots$

275. Recurring decimals.—When we attempt to reduce a fraction to a decimal, the process of division may or may not terminate. This is indicated at the right. When the divisors are 2, 4, 5, 8, 10, the division terminates. On the other hand, in dividing 1 by 3 the division never terminates, but the number 3 recurs constantly in the quotient. The fraction 1/6 reduces to a decimal in which the first figure is 1, and then 6 recurs constantly. In 1/7 we have the group 142857 and then this group is repeated. If the division were continued, this group would be repeated constantly. In 1/11 the recurring group is 09, and in 1/12 it is 3. Note that the recurring group may be preceded by figures different from those of this group. Such decimals are called recurring decimals.

$$\frac{1}{2} = .500000000000$$
$$\frac{1}{3} = .333333333333$$
$$\frac{1}{4} = .250000000000$$
$$\frac{1}{5} = .200000000000$$
$$\frac{1}{6} = .166666666666$$
$$\frac{1}{7} = .142857142857$$
$$\frac{1}{8} = .125000000000$$
$$\frac{1}{9} = .111111111111$$
$$\frac{1}{10} = .100000000000$$
$$\frac{1}{11} = .090909090909$$
$$\frac{1}{12} = .083333333333$$

We may now prove the theorem:

Every rational fraction may be reduced to a terminating or a recurring decimal.

This theorem may be proved by showing *why* any given fraction may be so reduced. Let us take 5/7. We start dividing 5.0 by 7 and note the successive remainders. When we come to the remainder 5 we have exactly the situation we had at the beginning, and the figures in the decimal must be repeated. In fact, the first time that any one figure is repeated as a remainder, the figures in the quotient will repeat from that point. Moreover, in dividing by 7 every remainder must be less than 7. Hence after six steps at the most, the first figure must be repeated, and therefore any fraction with denominator 7 reduces to a recurring decimal with at most six figures in the groups. In this manner we may show that a fraction m/n reduces to a terminating decimal, or to a recurring decimal with at most $n - 1$ figures in the groups. The groups in the recurring decimal may be preceded by a certain number of figures not belonging to any group.

$$7\overline{)5.00000000}\quad .71428571$$

remainders are:
1 3 2 6 4 5

We now note that:

Any recurring decimal may be regarded as an infinite geometric series.

Thus, $.3333333, \ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$ in which $a = \frac{3}{10}$ and $r = \frac{1}{10}$.

Thus the sum is $\dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}$, which for other reasons we also know equals the given decimal.

Consider now the recurring decimal $.37264264264 \ldots$ This equals $.37 + \frac{264}{10^5} + \frac{264}{10^8} + \frac{264}{10^{11}} + \ldots$

Omitting .37 for the present, we see that this is a geometric series in which $a = \frac{264}{10^5}$ and $r = \frac{1}{10^3}$.

Then the sum is $\frac{264}{10^5} \div \left(1 - \frac{1}{10^3}\right) = \frac{264}{10^5} \div \frac{1000 - 1}{10^3} = \frac{264}{10^2 \times 999} = \frac{264}{99900} = \frac{22}{8325}$, and $\frac{37}{100} + \frac{22}{8325} = \frac{12409}{33300}$, which is the value of this recurring decimal.

This shows that the number of figures in the groups of a recurring decimal may be small, though the fraction contains quite large numbers.

We may note that if an infinite decimal is not recurring in the manner described above, then it cannot be represented exactly by an ordinary fraction. Thus $\sqrt{2} = 1.4142 \ldots$ is a decimal of this kind that cannot be represented by a fraction of the type m/n.

EXERCISES

1. Reduce 4/13 to a recurring decimal. Check by finding the sum of the resulting series.
2. Reduce the recurring decimal .8888 . . . to a common fraction.
3. Reduce .2414141 . . . to a common fraction.
4. Reduce 5/17 to a recurring decimal. Check by finding the sum of the resulting series.
5. Reduce 4.61272727 . . . to a whole number and a common fraction.

276. Convergence of the series $1 + x + x^2 + \ldots$.—This is a geometric series in which $a = 1$, $r = x$, and the sum of the n terms is

$$\frac{1}{1-x} - \frac{x^n}{1-x}.$$

If $-1 < x < 1$, then $\frac{x^n}{1-x}$ approaches 0 as n becomes infinite, and hence the sum of the infinite series is $\frac{1}{1-x}$ for all values of x between -1 and 1.

$$S = \frac{1}{1-x}$$

By using for x the values ½, ⅔, ¾, ⅞, .9, .99, .999, .9999, we get the sums shown at the right. It is evident that as x grows larger, but remains less than 1, the sum of the series grows as large as we please.

As x approaches 1 the sum passes beyond all bounds, and ceases to "exist" the moment x becomes 1.

When x is negative, the sums behave as described in §273. By substituting for x the values $-½, -¾, -.9, -.99, -.9999$ we get sums shown at the right. All the sums are positive and greater than ½. But as x goes toward -1, the sums approach more and more nearly to ½. Note that $\frac{10000}{19999}$ is only $\frac{1}{39998}$ greater than ½.

Value of x	Sum of Series
1/2	2
2/3	3
3/4	4
7/8	8
.9	10
.99	100
.999	1000
.9999	10000
. . .	. . .
−1/2	2/3
−3/4	4/7
−.9	10/19
−.99	100/199
−.999	$\frac{10000}{19999}$
. . .	. . .

When $x = -1$ the formula gives $\frac{1}{1+1} = ½$. However, the series does not have a sum for this value of x. (See page 307.)

It is interesting to note that ½ is the average of the two values 1 and 0 taken alternately by the partial sums when $x = -1$. See page 307.

If in $S_n = \frac{x}{1-x} - \frac{x^n}{1-x}$ $(x < -1)$ then x^n increases in absolute value as n grows large and $\frac{x}{1-x}$ is farther and farther away from the sum of any fixed number of terms. The "steps" shown on page 305 increase indefinitely and no idea of sum can be connected with the series.

EXERCISES

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

1. Divide 1 by $1 - x$. Compare the quotient with the series studied in §276. After n steps, what will be the remainder? Study this remainder (a) for x between -1 and 1, (b) for $x = 1$ and $x = -1$, (c) for x greater than 1, (d) for x less than -1. Note that $1/(1 - x)$ has no meaning for $x = 1$.

To study the meaning of the remainder we must remember that it still remains to divide it by $1 - x$.

2. Divide 1 by $1 + x$ and study the remainder as under exercise 1. Compare the series $1 + x + x^2 + x^3 + \ldots$ for $x = \frac{1}{2}$, and the series $1 - x + x^2 - x^3 + \ldots$ for $x = -\frac{1}{2}$. Also compare these series for $x = \frac{3}{4}$ in the first series and $x = -\frac{3}{4}$ in the second.

3. Find the sum of the geometric series $a + ar + ar^2 + ar^3 + \ldots + ar^n = a(1 + r + r^2 + r^3 + \ldots + r^n)$ by multiplying it by $1 - r$. (See page 70 for the factors of $a^n - b^n$ or $1 - b^n$.)

4. In the series $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$ take out the factor $1 - x$ from each pair of successive terms, reducing it to $(1 - x)(1 + x^2 + x^4 + x^6 + \ldots)$. Find the sum of the series in the last parenthesis and multiply by $1 - x$. For what values of x is this series convergent? Compare the product with the sum of the original series.

5. Find the sum of the series $1 + x + x^2 + x^3 + x^4 + x^5$ by using the method used in exercise 4, and compare with the sum obtained by the usual method.

6. Describe the change in the value of $1/(1 - x)$ as x changes from 0 toward 1. If k is any fixed positive number how can we find a value of x, $x < 1$ for which $1/(1 - x) > k$?

7. Compare the value of $1/(1 - x)$ for $x = \frac{1}{2}$ and the sum of the infinite geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} +$ as found by the formula on page 306.

8. Find the ratio of each of the geometric series of which the following pairs of numbers are the first two terms. Then find the sum of each series to infinity.

(a) 5, 3	(e) 4, 1	(i) $3x, \frac{5}{2}x$	(m) $7x, -3x$
(b) 5, -3	(f) 9,4	(j) $4p, 3p$	(n) $2n, -\frac{1}{3}n$
(c) $-1, \frac{1}{2}$	(g) $a, \frac{1}{4}a$	(k) $1, -\frac{1}{4}$	(o) $-2y, \frac{1}{2}y$
(d) $1, -\frac{1}{2}$	(h) $p, \frac{1}{5}p$	(l) $2a, \frac{2}{3}a$	(p) $-9p, 7p$

277. Convergence or divergence of infinite series in general.—The sum of a finite series $a_1 + a_2 + \dots + a_n$ is often denoted by $\sum_{i=1}^{i=n} a_i$. In this expression i takes in order the values 1, 2, . . . , n of the subscripts in the terms of the series. The symbol Σ stands for "sum." The meaning of the whole symbol $\sum_{i=1}^{i=n} a_i$ is, then, that we start with the first term of the series, for which $i = 1$, and add the terms as the subscripts take all integral values up to n. The symbol is read "summation of a_i from $i = 1$ to $i = n$." This symbol is in universal use in mathematical writing. We have also used S_n to represent the same sum.

We now adopt the following general definition.

If for an infinite series $a_1 + a_2 + \dots + a_n + \dots$ *there exists a finite number* S *(sum) such that the limit of the sum of the first* n *terms approaches* S *as* n *becomes infinite, then the series is convergent.* Otherwise it is divergent.

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} a_i = S$$

The expression at the right is used to express this condition for convergence. More especially, this condition means that for the series in question there is one fixed number S such that for any fixed positive number d, however small, there is a positive number N such that the sum of the first n terms of the series differs from S by less than d for every n that is equal to or greater than N. This is expressed by the statement at the right, where n is *any* integer greater than N.

$$\left| \sum_{i=1}^{n} a_i - S \right| < d, \; n \geqq N$$

The idea described above is fundamental, and we shall illustrate by means of examples.

We know that the series $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots$ is convergent and that its sum is 2. (See §270.) If we specify a difference, such as $d = .00001$, how many terms of the series must we take so that their sum will differ from 2 by less than this difference? For this series we can actually find this number since we know that for any value of n the sum differs from 2 by $1/2^n$. We can then make a table of such differences for different values of n, as is indicated in the exercises on the next page.

EXERCISES

1. Check the differences, or remainders, given at the right.

n	$\frac{1}{2^n}$
1	.5
2	.25
3	.125
4	.0625
5	.03125
6	.015625
7	.0078125
8	.00390625
9	.001953125
10	.0009765625

2. If in the difference $\frac{1}{2^n}$, $n = 4$, the remainder will be less than .1. Will this statement be true for any value of n greater than 4?

3. In the same series, what must be the value of n so that the difference will be less than .01? What must be the value of n so that the difference will be less than .001? less than .0001?

278. Ways in which an infinite series may be divergent.—An infinite series may fail to be convergent in two ways:

1. The sum of the first n terms may become either plus or minus infinity as n becomes infinite. That is, the partial sums may grow large in either one of these directions.

2. The sum may approach more than one value as n becomes infinite.

Thus the sum of $1 + 1 + 1 + \ldots$ becomes $+\infty$ as n becomes infinite, while the sum $-1 - 1 - 1 \ldots$ becomes $-\infty$ as n becomes infinite.

On the other hand the sum of $1 - 1 + 1 - 1 + 1 - 1 \ldots$ is alternately 1 and 0 and hence both of these values are "approached."

Again, in $1 - 2 + 3 - 4 + 5 - \ldots$ the partial sums take values as at the right. Check these values of the partial sums. What values are approached by these sums as n becomes infinite?

$S_1 = 1$	$S_2 = 1 - 2 = -1$
$S_3 = -1 + 3 = 2$	$S_4 = 2 - 4 = -2$
$S_5 = -2 + 5 = 3$	$S_6 = 3 - 6 = -3$
.	.
.	.
.	.
$S_{2n-1} = n$	$S_{2n} = -n$

In $2 - 2^2 + 2^3 - 2^4 + 2^5 \ldots$ the successive sums are 2, −2, 6, −10, 22, −42, 86, . . . and we find that $+\infty$ and $-\infty$ are values approached.

All the series in these examples fail to be convergent. That is, they are all divergent.

279. ***Tests for convergence or divergence of infinite series.***—In the geometric series, which we have studied mainly thus far, we can find a formula giving the sum of the first n terms of the series and also the "remainder" or "difference" between the sum of n terms of the series and its total sum in case the series is convergent. These formulas enable us to decide whether the series is convergent or divergent.

But for most of the infinite series with which we have to work in mathematics no such formulas can be found, and we need other tests by means of which we can decide their convergence or divergence.

Infinite series are exceedingly useful in many kinds of practical computation and also in mathematical theory, but they cannot be used unless they are known to be convergent. Hence we need more general tests for convergence. Some of these we now proceed to study. In many cases the tests are obvious though rigorous proofs are complicated.

280. ***A necessary condition[1] for convergence of a series.***—*Theorem. If a series is convergent, then the absolute value of the* n*th term approaches zero as* n *becomes infinite.*

This theorem is verified by all the convergent series we have thus far considered. If S is the sum of the series, it is easily seen that if the terms do not grow small indefinitely then the sum S_n must differ from S for some values of n greater than any fixed number N, by an amount greater than d, if d is made small enough.

$$\text{If } \lim_{n \to \infty} \sum_{i=1}^{n} a_i = S, \quad \text{then } \lim_{i \to \infty} a_i = 0$$

It follows from this theorem that if the nth term of a series does not approach zero as n grows large, then the series cannot be convergent. This holds whether the terms are all positive or are in part positive and in part negative or are all negative.

However, the converse of this theorem does not hold. That is, the terms may approach zero as n grows large, and still the series may be divergent. We shall now give an example of such a series.

[1] The expression "necessary condition" is much used in mathematics. It means that the condition is "necessary" for a theorem. It can be proved to be a logical consequence of the theorem, as in the above instance.

281. The harmonic series.—The series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

is called the harmonic series, since its terms are the reciprocals of the arithmetic sequence 1, 2, 3, . . . , n, . . . (See §204.) Obviously the nth term, $1/n$, approaches zero as n grows large. But that this series is divergent may be shown as follows.

$$\begin{aligned} \frac{1}{3}+\frac{1}{4} &> \frac{1}{2} \\ \frac{1}{5}+\ldots+\frac{1}{8} &> \frac{1}{2} \\ \frac{1}{9}+\ldots+\frac{1}{16} &> \frac{1}{2} \\ \frac{1}{17}+\ldots+\frac{1}{32} &> \frac{1}{2} \\ \frac{1}{33}+\ldots+\frac{1}{64} &> \frac{1}{2} \\ \ldots\ldots\ldots \end{aligned}$$

In the sum of the 3d and the 4th terms, or $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{3}$ is greater than $\frac{1}{4}$ and hence $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Again, in the sum of the 5th to the 8th terms, or $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$, the terms $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ are all greater than $\frac{1}{8}$, and hence $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$. In this way we can build an endless sequence of inequalities such as those given at the right. Hence by taking enough terms we can make the sum exceed $k\cdot\frac{1}{2}$ for any value of k however large, or as many times $\frac{1}{2}$ as we please, and the sum becomes infinite as n grows large indefinitely. That is, the harmonic series is divergent.

282. The comparison test for convergence.—*Theorem. If* $a_1+a_2+a_3+\ldots$ *and* $b_1+b_2+b_3+\ldots$ *are series of positive terms and if* (1) $a_1+a_2+a_3+\ldots$ *is convergent, and* (2) *after a certain term* $b_i<a_i$ *for all values of* i, *then* $b_1+b_2+b_3+\ldots$ *is convergent.*

This theorem follows directly from the fundamental proposition in §272.

EXERCISES

1. State the fundamental proposition in §272, and illustrate by a sequence of numbers. Does the sequence $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, \ldots$ illustrate this proposition? Make other such sequences.

2. Carry out in detail a proof of the theorem in §282.

3. State and prove a theorem similar to that in §282 if all the terms in both series are negative.

283. *Comparison test for divergence.*—The following is closely related to the theorem in §282.

Theorem. If $a_1 + a_2 + a_3 + \ldots$ *and* $b_1 + b_2 + b_3 + \ldots$ *are both series of positive terms, if* $a_1 + a_2 + a_3 + \ldots$ *is divergent, and if after a certain term* $b_i > a_i$ *for all values of* i, *then* $b_1 + b_2 + b_3 + \ldots$ *is divergent.*

Thus we know that the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$ is divergent since its terms are greater than the corresponding terms in the harmonic series.

284. *A fundamental property of convergent series.*—The formula for the sum of an infinite geometric series gives a "remainder" $\frac{ar^n}{1-r}$ in case we stop adding terms after reaching the *n*th term. By taking *n* large enough, this "remainder" can be made as small as we please.

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

We shall now state a property closely connected with the above, which holds for every convergent series, even when we have no formula giving the sum of *n* terms or of the whole series.

Theorem. (1) *If a series* $a_1 + a_2 + a_3 + \ldots a_n + \ldots$ *is convergent, then for every positive number* d, *however small, there is a number* n_1 *such that the sum of any number of terms after the* n_1*th term is less than* d *in absolute value.*

(2) [Converse of (1)] *A series* $a_1 + a_2 + \ldots + a_n + \ldots$ *is convergent if for every positive number* d *there is a number* n_1 *such that the sum of any number of terms after the* n_1*th is less than* d *in absolute value.*

It is rather easy to see why this theorem must hold, but a rigorous proof is somewhat complicated, and will not be given here. Instead we shall try to make clear just what the theorem means.

Consider a convergent infinite series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$. Suppose a small positive number d_1 is given. Then the theorem asserts that there is a certain term a_{n_1} in the series such that the sum of any number of terms beyond a_{n_1} is less than d_1 in absolute value. The absolute value restriction is put in because the sum of these terms may be negative and hence less than d_1 no matter how large its absolute value may be.

If now a smaller positive number d_2 is given, then there is another term a_{n_2}, such that the sum of any number of terms following it will be less than d_2. This process may be continued for an infinite sequence of d's approaching zero in value.

The difference between this and the treatment by means of the "remainder" in the geometric series is that we are here taking any part of the "remainder." Note the "sum of any number of terms following the term a_{n_1}."

When part (1) of this theorem is clearly understood, the meaning of its converse, part (2), will be clear.

By means of this general theorem we can now prove the following more general, and also more useful, form of the comparison test given in §282.

285. A more general comparison test.—*Theorem. If a series of positive terms,* $a_1 + a_2 + \ldots a_n + \ldots$, *is convergent and if in a series* $b_1 + b_2 + \ldots + b_n + \ldots$

$$|b_1| \leqq a_1,\ |b_2| \leqq a_2,\ \ldots,\ |b_n| \leqq a_n,\ \ldots$$

then $b_1 + b_2 + \ldots + b_n + \ldots$ *is convergent.*

To prove this theorem we need to make use of the fact that $|b_1 + b_2 + \ldots + b_n| \leqq |b_1| + |b_2| + \ldots + |b_n|$. If all terms $b_1, b_2, \ldots, b_n$ are positive, or all negative, then the equality sign holds. If some of these terms are positive and some negative, then the "less than" ($<$) sign holds. These statements are easily verified.

To prove our main theorem we now notice that since $a_1 + a_2 + \ldots$ is convergent then by (1) of the theorem in §284 there is an n_1 such that the sum of any number of terms beyond a_{n_1} is less than d. But the sum of the corresponding terms in $b_1 + b_2 + \ldots$ is equal to or less than this sum and hence $b_1 + b_2 + \ldots$ is convergent by (2) of the theorem in §284.

EXERCISES

1. Illustrate the statement $|b_1 + b_2 + \ldots + b_n| \leqq |b_1| + |b_2| + \ldots + |b_n|$ by giving numerical values to the b's.

2. Carry out in detail a proof of the theorem in §283. Compare with the proof of the theorem in §282.

286. ***Alternating series; series of mixed terms.***—A series in which terms are alternately positive and negative is called an alternating series. If a series has an infinite number of positive terms and also an infinite number of negative terms, it is called a series of mixed terms. Obviously an alternating series is a series of mixed terms, while a series of mixed terms may or may not be alternating.

The following is a direct consequence of §285.

Theorem. If in a convergent series of positive terms the signs of any number of terms are changed, the result is a convergent series.

For, clearly, the change of signs of some or all of the terms cannot increase the absolute value of the sum of any given set of terms.

This proposition is not true for a series of mixed terms. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \text{ is convergent, while}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \text{ is divergent. } (\S 281)$$

287. ***Absolutely convergent series.***—A series is said to be absolutely convergent if the series formed by making all its terms positive is convergent. Otherwise it is said to be conditionally convergent.

Thus the series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$ is absolutely convergent since $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ is convergent. For an example of a conditionally convergent series see exercise 1, p. 319.

It is corollary of §286 that an absolutely convergent series is convergent, while the converse of this proposition is not true.

288. ***Test for convergence of alternating series.***—*Theorem. An alternating series is convergent if (a) its* n*th term approaches zero as* $n \to \infty$, *and if also (b) after a certain point the absolute value of each term is less than that of the term immediately preceding it.*

PROOF. Suppose we start with any positive term in this series. Then with a process exactly like that used in §273, it is evident that the absolute value of the sum of any number of terms following is less than this term.

That is, we have steps in alternate directions, each step shorter than the preceding, and hence no matter how long we continue we shall never be farther from the starting point than the first step. Since we can select a starting point so that the first step will be as small as we please, it follows by §284 that the series is convergent.

That both of the conditions (a) and (b) in this theorem are necessary in order that the series will be convergent is easily seen. By §280 the terms must approach zero. That (b) is necessary is shown by the following example.

Consider the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ (see §281) and the convergent series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$. Change all the signs in the second series and then sandwich its terms in alternately with the terms of the first series, forming the series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{4} - \frac{1}{2^3} + \frac{1}{5} - \frac{1}{2^4} + \ldots .$$

This series is alternating and its terms approach zero as the number of terms increase. But each term is not numerically smaller than the one preceding it. Thus 1/5 is not smaller than $\left|\frac{1}{2^3}\right| = \frac{1}{8}$. It is easily seen that this series is divergent.

EXERCISES

1. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ is a convergent series. Use §288. Is this series absolutely or conditionally convergent? Use §281 in proving your answer.

2. Show in detail that the series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{4} - \frac{1}{2^3} + \ldots$$

used above is divergent.

3. If $a_1 + a_2 + \ldots + a_i + \ldots$ and $b_1 + b_2 + \ldots b_i + \ldots$ are both divergent series, can you make any statement as to the divergence or convergence of the sum $a_1 - b_1 + a_2 - b_2 + \ldots a_i - b_i + \ldots$? If so, what?

289. Convergence of a part of a convergent series.—*Theorem. If a series is convergent, then a series formed by omitting or inserting a finite number of terms is a convergent series.*

This theorem is obvious. Let S be the sum of the given series, and let S_1 be the sum of the terms omitted or added. Then the sums of the new series will be $S - S_1$ and $S + S_1$ respectively.

The tests for convergence specify that a certain condition must be satisfied *after* a certain term. This means that the series beginning with this term is proved convergent. Hence by the present theorem the whole series will be convergent.

290. Multiplying or dividing the terms of an infinite series.—The following theorem is obvious.

If the terms of an infinite series are multiplied or divided by a fixed number different from zero, then the first series and the resulting series are both convergent or both divergent.

That is, if series (1) at the right is convergent, then (2) and (3) are convergent; and if (1) is divergent, then (2) and (3) are divergent.

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \quad (1)$$
$$ka_1 + ka_2 + ka_3 + \ldots + ka_n + \ldots \quad (2)$$
$$\frac{a_1}{k} + \frac{a_2}{k} + \frac{a_3}{k} + \ldots + \frac{a_n}{k} + \ldots \quad (3)$$

291. Summary of convergence tests studied thus far.—The tests for convergence or divergence of series studied thus far are:

1. The limit of the nth term must be zero. (§280)
2. Comparison test for convergence. (§282)
3. Comparison test for divergence. (§283)
4. Test for convergence of alternating series. (§288)

Example 1. Write the general or nth term of the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$. Decide whether this series is convergent or divergent.

Solution: The nth term is $\frac{1}{2n-1}$. By §281 we know that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent. Hence by §290 $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots$ is divergent. Therefore it follows from §283 that $1 + \frac{1}{3} + \frac{1}{5} + \ldots$ is divergent.

Example 2. Write the nth term of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$ . . . Test this series for convergence.

SOLUTION: The nth term is $(-1)^{n-1}\frac{1}{\sqrt{n}}$. The factor $(-1)^{n-1}$ is used to indicate the sign of the term. Note that $(-1)^{n-1} = +1$ when n is odd (the exponent $n - 1$ is even), and $(-1)^{n-1} = -1$ when n is even.

This series has properties (a) and (b) stated in the theorem of §288. Hence the series is convergent.

Example 3. Write the nth term of the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{33} + \cdots$. Test this series for convergence.

SOLUTION: The nth term is $\frac{1}{2^n + 1}$. The series is proved convergent by comparing it with the convergent series whose nth term is $1/2^n$.

Show that $\frac{1}{2^n + 1} < \frac{1}{2^n}$.

Example 4. Write the general term of the series $\frac{1^2 + 1}{1^3 + 1} + \frac{2^2 + 1}{2^3 + 1} + \frac{3^2 + 1}{3^3 + 1} + \cdots$.

SOLUTION: The nth term is $\frac{n^2 + 1}{n^3 + 1}$. By reducing $\frac{n^2 + 1}{n^3 + 1}$ and $\frac{1}{n}$ to a common denominator it is seen that for $n > 1$, $\frac{n^2 + 1}{n^3 + 1} > \frac{1}{n}$. Hence by comparison with the harmonic series it follows that the given series is divergent.

EXERCISES

Write the nth term in each of the following series. Then determine whether the series is convergent or divergent.

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$
2. $1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \cdots$
3. $1 + \frac{1}{2^3} + \frac{1}{2^5} + \cdots$
4. $\frac{2^3 + 1}{2^4 + 1} + \frac{3^3 + 1}{3^4 + 1} + \cdots$
5. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$
6. $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$
7. $\frac{2^4 + 1}{2^5 + 1} + \frac{3^4 + 1}{3^5 + 1}$
8. $1 - \frac{1}{\sqrt[5]{2}} + \frac{1}{\sqrt[5]{3}} - \frac{1}{\sqrt[5]{4}} + \cdots$

292. Ratio test for convergence of series.—The following is called the ratio test for convergence of series.

Theorem. If after a certain term in an infinite series the ratio $\frac{a_{n+1}}{a_n}$ *lies between two fixed numbers which are between* -1 *and* 1, *then the series is convergent.*

$$-1 < k_1 < \frac{a_{n+1}}{a_n} < k_2 < 1$$

PROOF: Let k_1 and k_2 be two numbers such that $-1<k_1<k_2<1$, and let r be the greater of the two absolute values $|k_1|$ and $|k_2|$. Then r is positive and less than 1. Let n_1 be the value of n after which the given ratio lies between k_1 and k_2. Suppose that a_{n_1} is positive. Then the series

$$a_{n_1} + a_{n_1}r + a_{n_1}r^2 + \ldots \qquad (1)$$

is a convergent geometric series of positive terms.

But after the term a_{n_1} the absolute values of the terms of the given series are numerically less than the corresponding terms of series (1). Hence by §285 the given series is convergent after the n_1th term, and by §289 the whole series is convergent.

Note that it is not sufficient that the ratio in this theorem lies between -1 and 1. It must be kept from approaching these by being made to lie between two fixed numbers that lie between -1 and 1.

293. The p*-series.*—We shall now prove the following theorem. *If* p *is greater than* 1, *then the series*

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots \textit{ is convergent.}$$

PROOF: The proof depends upon the fact that $\frac{1}{3^p} < \frac{1}{2^p}$; $\frac{1}{5^p}, \frac{1}{6^p}, \frac{1}{7^p}$, are all less than $\frac{1}{4^p}$; and so on. Then it follows that

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p},$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p},$$

as indicated at the right. Hence the sum of the series is less than the sum of

$$\frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \frac{16}{16^p} + \cdots$$

or

$$\frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \frac{1}{16^{p-1}} + \cdots \quad (1)$$

$$\begin{aligned} \frac{1}{2^p} + \frac{1}{3^p} &< \frac{2}{2^p} \\ \frac{1}{4^p} + \cdots + \frac{1}{7^p} &< \frac{4}{4^p} \\ \frac{1}{8^p} + \cdots + \frac{1}{15^p} &< \frac{8}{8^p} \\ \frac{1}{16^p} + \cdots + \frac{1}{31^p} &< \frac{16}{16^p} \end{aligned}$$

But (1) is a geometric series in which the ratio is $1/2^{p-1}$. Since $p > 1$, $p - 1$ is positive and 2^{p-1} is greater than 1. Hence series (1) is convergent, and by the fundamental principle of §272 the given series is convergent.

This series, called the p-series, is often convenient for use in making a comparison test for the convergence of a series.

REMARK: We know (§281) that the p-series is divergent for $p = 1$. It is easily proved that it is divergent for all values of p that are less than 1. But for every value of p greater than 1, we have just proved it convergent. It is clear that when the value of p is made large, the terms of this series, and hence its sum, grow smaller. Thus for $p = 2$ we have $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$, while for $p = 4$ the series is $1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \cdots$. On the other hand, when p is "near" 1, then the terms of the series are nearer the terms of the harmonic series and the sum is large.

If we think of p as taking a sequence of values approaching 1 as a limit, then the sums become gradually larger, and increase without bound.

It is interesting to note that the terms of a series may be smaller than the terms of the harmonic series and still be divergent. That is, a series cannot be proved convergent by comparing its terms with those of the harmonic series. Similarly, a series cannot be proved divergent by comparing its terms with those of the p-series.

The following is easily seen. If $a_1 + a_2 + \cdots$ is convergent then it is possible to find a convergent series $b_1 + b_2 + \cdots$ such that $b_i > a_i$; and if the a series is divergent, a divergent b series may be found such that $b_i < a_i$.

294. Series in which the ratio of consecutive terms approaches unity.—Compare the two series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots \qquad (1)$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}+\ldots \qquad (2)$$

In series (1), the ratio of consecutive terms is $\frac{1}{n+1}\div\frac{1}{n}=\frac{n}{n+1}$. As n increases indefinitely this ratio approaches 1. In series (2), the ratio of consecutive terms is $\frac{1}{(n+1)^2}\div\frac{1}{n^2}=\frac{n^2}{(n+1)^2}=\left(\frac{n}{n+1}\right)^2$. As n increases indefinitely, this ratio also approaches 1. But we know that series (1) is divergent while series (2) is convergent. Hence the fact that the ratio approaches unity does not serve as a test for either divergence or convergence. The same is also true when the ratio lies between 0 and -1 but approaches -1 as n increases indefinitely. The Examples in §295 will illustrate.

295. Finding the limit of a fraction.—Applying the ratio test for convergence of a series sometimes requires finding the limit of a fraction. Examples will show how this may be done.

Example 1. Find the limit of $\frac{2n^2-3n+6}{3n^2+5n-3}$ as n increases indefinitely.

SOLUTION: $\frac{2n^2-3n+6}{3n^2+5n-3}=\dfrac{2-\frac{3}{n}+\frac{6}{n^2}}{3+\frac{5}{n}-\frac{3}{n^2}}$, whose limit is $\frac{2}{3}$.

The terms of the fraction are divided by n^2, when the limit is easily seen to be $\frac{2}{3}$ as n increases indefinitely.

Example 2. Test the series $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$.

SOLUTION: The nth and $(n+1)$st terms are $\frac{n}{2^n}$ and $\frac{n+1}{2^{n+1}}$ and their ratio is $\frac{n+1}{2^{n+1}}\div\frac{n}{2^n}=\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}=\frac{n+1}{n}\cdot\frac{1}{2}$. But $\lim\limits_{n\to\infty}\frac{n+1}{n}=1$. Hence the limit of the ratio is $\frac{1}{2}$. Since the ratio is "near" $\frac{1}{2}$ it follows by §292 that the series is convergent.

Example 3. Test the series $\frac{2}{3}\cdot\frac{1}{1}+\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}$ for convergence.

SOLUTION: By §290, the series $\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{4}$ or $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots$, is divergent. Then by comparison, the given series is divergent.

Example 4. Test the series $\frac{2}{2\cdot 3}+\frac{4}{3\cdot 4}+\frac{6}{4\cdot 5}+\ldots+\frac{2n}{(n+1)(n+2)}+\ldots$ for convergence.

SOLUTION: Compare the nth term in this series with the nth term in the harmonic series. Reduce the last two terms to a common denominator and compare the numerators. For any value of n greater than 2, the first numerator is greater than the second and hence the given series is divergent. (§283.)

$$\frac{2n}{(n+1)(n+2)} \qquad \frac{1}{n}$$

$$\frac{2n^2}{n(n+1)(n+2)}, \qquad \frac{n^2+3n+2}{n(n+1)(n+2)}$$

EXERCISES

Write the nth term in each of the following series and test it for convergence.

1. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots$
2. $\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\ldots$
3. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$
4. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$
5. $1-\frac{2}{\sqrt{2}}+\frac{3}{2\sqrt{3}}-\frac{4}{3\sqrt{4}}+\ldots$
6. $\frac{1}{\log 2}+\frac{1}{\log 3}+\ldots$
7. $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\ldots$
8. $\frac{1\cdot 2}{3\cdot 4\cdot 5}+\frac{2\cdot 3}{4\cdot 5\cdot 6}+\frac{3\cdot 4}{5\cdot 6\cdot 7}+\ldots$

SUGGESTION FOR EXERCISE 8: Multiply the terms of this series by 2. Then the given series is convergent or divergent according as this series is convergent or divergent.

Compare the nth term $\frac{2n(n+1)}{(n+2)(n+3)(n+4)}$ with the nth term $\frac{1}{n}$ of the harmonic series. To do this reduce these fractions to a common denominator as in Example 4 above.

9. $\frac{1}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\frac{3}{4\cdot 5\cdot 6}+\ldots$
10. $\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{3}{4\cdot 5}+\ldots$

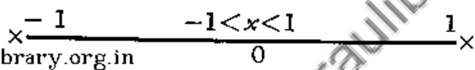
SUGGESTIONS: In exercise 9, compare the nth term with $1/n^2$.
In exercise 10, multiply the nth term by 2 and compare with $1/n$.

296. Series of functions; interval of convergence.—The terms of a series may be functions of a variable instead of fixed numbers. We have already studied one such series, namely,

$$1 + x + x^2 + \ldots + x^n + \ldots \qquad (1)$$

When we substitute a definite number for x, we have an ordinary series such as we have now studied in detail. The only new questions raised by a series of functions is as to what values of the independent variable will make the series convergent. We have already seen (page 310) that series (1) is convergent for any value of x between -1 and 1. That is, this is the interval of convergence of this series.

The interval of convergence may be represented geometrically.



In the figure the heavy line represents the interval of convergence of the above series. The little crosses, × ×, at the ends of the line indicate that these points are not included in this interval. That is, the series is convergent, not for $x = -1$, or for $x = 1$, but for all values of x between these.

Example 1. Find the interval of convergence of the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots$$

Solution: The ratio of the $(n + 1)$st and the nth terms is

$$\frac{(n+1)x^n}{nx^{n-1}} = \frac{n+1}{n}x.$$

The fraction $\frac{n+1}{n}$ is greater than 1 but can be made as nearly equal to 1 as we please by taking n sufficiently large. Hence it is obvious that the series is divergent for $x = 1$, $x = -1$.

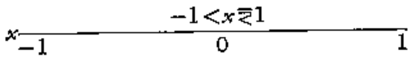
Let x_1 be any value of x between -1 and 1, let k_1 and k'_1 be numbers such that $-1 < k_1 < x_1 < k'_1 < 1$. Then by taking n sufficiently large, $\frac{n+1}{n}x_1$ can be made to lie between k_1 and k'_1 and hence by §292 the series is convergent. That is, the required interval of convergence is $-1 < x < 1$. This interval is represented in the above figure.

In the same manner we may find the interval of convergence of $1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$. What is this interval?

Example 2. Find the interval of convergence of the series

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \ldots$$

We know (§281) that this series is divergent for $x = -1$ and convergent for $x = 1$ (§286). That this series is convergent for x between -1 and 1 follows immediately from the theorem in §297. That is, the interval of convergence is $-1 < x \leqq 1$.



The cross at $x = -1$ in the figure indicates that -1 is not included in the interval of convergence, while the absence of the cross at $x = 1$ indicates that this point is included.

297. *The power series.*—A series of the general form

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

is called a power series in x. The series of functions that we have studied are all of this type. We shall now state and prove a general theorem on the convergence of such series.

Theorem. If a power series in x *is convergent for a certain value* x_1 *of* x, *then it is convergent for every value of* **x** *whose absolute value is less than that of* x_1.

If the series is divergent for $x = x_1$, *then it is divergent for all values of* x *whose absolute value is greater than that of* x_1.

PROOF: Since the series $a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + \ldots + a_nx_1^n + \ldots$ is convergent, the ratio at the right must be less than 1 in absolute value, though it may approach 1 as $n \to \infty$. But for $|x_2| < |x_1|$, this ratio not only lies between -1 and 1, but cannot approach either of these values. Hence this ratio lies between two fixed numbers which themselves lie between -1 and 1. Therefore by §292 the series is convergent for every such value of x. The proof of the second part of the theorem is now obvious. This theorem is used in Example 2 above where it enables us to determine at once the exact interval of convergence since we know that the series is divergent for $x < -1$ and $1 < x$, and also that it is divergent for $x = -1$ and convergent for $x = 1$.

$$\boxed{\frac{a_{n+1}x_1^{n+1}}{a_nx_1^n} = \frac{a_{n+1}}{a_n} \cdot x_1}$$

298. Examples in convergence or divergence of power series.—

Example 1. Find the interval of convergence of the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

SOLUTION: The ratio of the $(n + 1)$st term and the nth term is:

$$\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!} = \frac{x}{n}.$$

Since for any fixed value of x this ratio approaches zero as n approaches infinity it follows that the series is convergent for every value of x. Hence we say that the interval of convergence is $-\infty < x < \infty$.

This series is called the exponential series (its value is e^x); it is said to be permanently convergent.

Example 2. Find the intervals of convergence of the series

$$1 + x + 1 \cdot 2x^2 + 1 \cdot 2 \cdot 3x^3 + \ldots + n!x^n + \ldots$$

SOLUTION: The ratio of the $(n + 1)$st and the nth term is

$$\frac{(n+1)!x^{n+1}}{n!x^n} = (n+1)x$$

For any fixed value of x, except $x = 0$, the absolute value of this ratio is greater than unity when n is taken sufficiently large. Hence the term cannot approach zero, and by §280 the series is divergent for every value of x except $x = 0$.

Such a series is said to be permanently divergent.

EXERCISES

Find the intervals of convergence of the following series.

1. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ (the cosine series)
2. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ (the sine series)
3. $1 - x + \frac{x^2}{2} - \frac{x^3}{3}$
4. $1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots$
5. $1 - 2x + 2^2x^2 - 2^3x^3 + \ldots + (-1)^{n-1}2^nx^n + \ldots$
6. $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \ldots + \frac{x^{n-1}}{a^{n-1}} + \ldots$
7. $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \ldots + (-1)^{n-1}\frac{x^{n-1}}{a^{n-1}} + \ldots$

299. The binomial expansion.—The expansion by the binomial theorem of $(a+b)^n$ terminates after the $(n+1)$st term in case n is a positive integer since all subsequent terms have a factor zero. If n is not a positive integer the expansion is an infinite series and then it becomes important to decide whether the series is convergent, since only in that case can the series be used for computation.

By §268, the rth and $(r+1)$st terms of this series are

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}$$

and
$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}b^{r},$$

and their ratio is

$$\frac{\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}b^{r}}{\dfrac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}} = \frac{n-r+1}{r}\cdot\frac{b}{a}.$$

For any fixed value of n, n not a positive integer, this ratio approaches $-\dfrac{b}{a}$ as r approaches infinity. Hence the series is convergent when b is numerically less than a.

The expression $(a+b)^n$ may be changed to $a^n\left(1+\dfrac{b}{a}\right)^n$ when the expansion reduces to the expansion of the form $(1+x)^n$, which is a power series in x.

Thus, $\sqrt{a^2+x^2} = a\sqrt{1+\left(\dfrac{x}{a}\right)^2} = a(1+u^2)^{\frac{1}{2}}$

$$= a\left[1+\frac{1}{2}u^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}u^4+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{2\cdot 3}u^6+\cdots\right.$$

$$= 1+\frac{1}{2}u^2-\frac{u^4}{8}+\frac{u^6}{16}+\cdots$$

which we can show to be convergent for $-1<u<1$.

MISCELLANEOUS EXERCISE IN INFINITE SERIES

Test the following series for convergence. A few of these series have been studied earlier.

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$

2. $\frac{1}{4} + \frac{2!}{4^2} + \frac{3!}{4^3} + \ldots$

3. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$

4. $\frac{1}{2} + \frac{2^2}{2^2} + \frac{2^2}{2^3} + \frac{2^2}{2^4} + \ldots$

5. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots$

6. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots$

7. $\frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \ldots$

8. $\frac{2}{1 \cdot 3} - \frac{2 \cdot 4}{1 \cdot 3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7} - \ldots$

9. $\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \ldots$

10. $\frac{1}{1\sqrt[3]{1}} + \frac{1}{2\sqrt[3]{2}} + \frac{1}{3\sqrt[3]{3}} + \ldots$

11. $\frac{1}{2+3} + \frac{1}{2+3^2} + \frac{1}{2+3^3} + \ldots$

12. $\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} +$

13. $\frac{1}{a+1^2} + \frac{1}{a+2^2} + \frac{1}{a+3^2}$, it being given that a is positive.

14. $\frac{1}{1+a} + \frac{1}{2+a} + \frac{1}{3+a}$, it being given that a is negative.

15. $\frac{1 \cdot 2}{2} + \frac{3 \cdot 4}{2^2} + \frac{5 \cdot 6}{2^3} + \ldots + \frac{(2n-1)2n}{2^n} + \ldots$

Find the interval of convergence of each of the following.

16. $1 + 3x + 5x^2 + 7x^3 + \ldots$

17. $\frac{1}{3 \cdot 4} + \frac{x}{5 \cdot 6} + \frac{x^2}{7 \cdot 8} + \ldots$

18. $x + \frac{3}{5}x^2 + \ldots + \frac{n^2-1}{n^2+1}x^n$

19. $\frac{2}{5}x + \frac{7}{10}x^2 + \ldots + \frac{2^n-2}{2^n+1}x^{n-1}$

20. $1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \ldots$

21. $1 + \frac{2x}{2!} + \frac{3^2x^2}{3!} + \frac{4^3x^3}{4!}$

22. $\frac{1}{2} + \frac{4}{7}x + \frac{6}{10}x^2 + \ldots + \frac{2n}{3n+1}x^n$

23. $\frac{1}{7} + \frac{1}{4}x + \ldots + \frac{n^2}{2n^2+3n+2}x^{n-1} + \ldots$

24. $\frac{1}{2}x + \frac{4}{9}x^2 + \frac{9}{28}x^3 + \ldots + \frac{n^2}{n^3+1}x^n + \ldots$

25. $\frac{1}{3}x + \frac{\sqrt{2}}{5+\sqrt{2}}x^2 + \ldots + \frac{\sqrt{n}}{n^2+\sqrt{n}+1}x^n + \ldots$

CHAPTER 24:

PARTIAL FRACTIONS

The problem to be studied in this chapter is in a sense the "inverse" of adding or subtracting ordinary algebraic fractions.

Thus, as shown at the right, we may add $\frac{1}{x-1}$

$$\frac{1}{x-1}+\frac{2}{x+3}=\frac{x+3+2x-2}{(x-1)(x+3)}=\frac{3x+1}{(x-1)(x+3)}$$

and $\frac{2}{x+3}$ and obtain the sum $\frac{3x+1}{(x-1)(x+3)}$. Our problem now is to start with $\frac{3x+1}{(x-1)(x+3)}$ and find the two fractions whose sum it is. It is an essential part of our problem that we are to find fractions with denominators of lower degree than that of the fraction with which we start. We speak of the process as resolving, or decomposing, a given fraction into partial fractions. We shall begin with a description of the kind of fractions we are to resolve into partial fractions.

300. *Proper fractions in their simplest form.*—The fractions with which we are to deal in this chapter are of the type shown at the right, in which $p_1(x)$ and $p_2(x)$ are ordinary polynomials in x. All common factors are removed from the numerator and denominator so that the fractions are in the "lowest terms." Also, in the fractions that we are to decompose, the numerators are of lower degree in x than the denominators. That is, the fractions are proper fractions in their lowest terms.

$$\frac{p_1(x)}{p_2(x)}$$

Improper fractions are reduced by division as in the example at the right.

$$\frac{3x^2-3x+6}{x^2+5x-3}=3-\frac{18x-15}{x^2+5x-3}$$

The common factor 3 of the terms of the numerator may then be removed and the remaining part of our problem is to decompose, if possible, the fraction at the right. In this case it turns out that such decomposition is not possible.

$$\frac{6x-5}{x^2+5x-3}$$

301. Form of fractions resulting from resolving a given fraction.—In adding fractions, the sum is a fraction whose denominator is the lowest common multiple of the denominators of the given fractions. Hence the denominators of the fractions into which a given fraction may be resolved must be such that their lowest common multiple is the denominator of the given fraction. In practice we resolve the fraction so that each denominator will have no real factors, with the exception that a first degree factor may be repeated in the same denominator.

That is, the denominators will be of the form $ax + b$, $(ax + b)^2$, . . . , and $ax^2 + bx + c$. We shall see later that any polynomial is the product of real factors of these types.

Also, the numerator of each fraction must be of lower degree than the denominator.

$$(1) \quad \frac{x+5}{(x+2)(x-3)(x+1)} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+1}$$

$$(2) \quad \frac{x^2+7x-1}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$(3) \quad \frac{x^2-2x+3}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}$$

Thus if we are to resolve (1) at the right, the resulting denominators will be $x + 2$, $x - 3$, and $x + 1$, and the numerators will be constants.

In resolving (2), the denominators will be $x^2 + x + 1$ and $x - 2$. The numerator of the first fraction will be of the form $Ax + B$ in which either A or B, but not both, may turn out to be zero.

In resolving (3) the denominators will be $(x - 1)^2$, $x - 1$, and $x + 2$. But why should not the numerator of $(x - 1)^2$ be of the form $Ax + B$ instead of A? This question is answered on page 334.

We shall now learn to carry out the process of resolving fractions into partial fractions, justifying the process by showing that it actually works. A complete mathematical justification is given on pages 340–342.

When the denominator is factored, the remaining part of this process consists in finding the values of the constants A, B, C that will make the sum of the partial fractions identically equal to the given fraction.

302. *Resolving fractions having denominators with unlike linear factors.*

Example 1. Resolve $\frac{x-4}{(x+2)(x-1)}$ into partial fractions. By the above, the form of the decomposition must be as at the right.

$$\frac{x-4}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Our problem then is to find values of A and B such that when the resulting fractions are added, the sum will be the given fraction. Adding the fractions gives $\frac{(A+B)x+2B-A}{(x+2)(x-1)}$. But this must be identically equal to the given fraction. Hence $A+B = 1$, $2B-A = -4$, and therefore $A = 2$, $B = -1$.

$$\frac{A}{x+2} + \frac{B}{x-1} = \frac{Ax - A + Bx + 2B}{(x+2)(x-1)} = \frac{(A+B)x + 2B - A}{(x+2)(x-1)}$$

$$A + B = 1$$
$$2B - A = -4$$
$$A = 2, \; B = -1$$

Therefore $$\frac{x-4}{(x+2)(x-1)} = \frac{2}{x+2} - \frac{1}{x-1}.$$

The values of A and B may be found more simply as follows.

$$\frac{x-4}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}.$$

Clearing of fractions, $x - 4 = A(x-1) + B(x+2)$.
But this is an identity in x and we may substitute any value of x that we please. Substituting in succession $x = 1$ and $x = -2$, we have $B = -1$, $A = 2$. This method will serve to resolve any fraction whose denominator is the product of different first degree factors. Note that $x = 1$, $x = -2$ are permissible substitutions in $x - 4 = A(x-1) + B(x+2)$ but not in the given fraction. See §§304, 306.

Example 2. Resolve $\frac{x^2-3x+1}{(x-1)(x+4)(x-2)}$ into partial fractions.

Solution: Let $\frac{x^2-3x+1}{(x-1)(x+4)(x-2)} = \frac{A}{x-1} + \frac{B}{x+4} + \frac{C}{x-2}.$

Clearing of fractions,

$$x^2 - 3x + 1 = A(x+4)(x-2) + B(x-1)(x-2) + C(x-1)(x+4).$$

Substituting $x = 1$ we see at once that the second member is reduced to $A(5)(-1) = -5A$, while the first member is reduced to -1. Hence $5A = 1$ and $A = 1/5$.
Substituting in order $x = -4$ and $x = 2$, we obtain $B = 29/30$ and $C = -1/6$.

Hence $$\frac{x^2-3x+1}{(x-1)(x+4)(x-2)} = \frac{1}{5(x-1)} + \frac{29}{30(x+4)} - \frac{1}{6(x-2)}.$$

Verify this solution by adding the fractions in the right member of this equation.

EXERCISES

Resolve the following into partial fractions. Check the first four by adding the resulting fractions. When the denominators are not given as the product of linear factors, the first step is to factor the denominator. Thus in exercise 1, $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$.

1. $\frac{1}{x^2-1}$
2. $\frac{1}{x^2-4}$
3. $\frac{1}{x^2-16}$
4. $\frac{a}{a^2x^2-b^2}$
5. $\frac{1}{x^2+6x+5}$
6. $\frac{1}{x^2-6x+8}$
7. $\frac{2x+7}{x^2+x-2}$
8. $\frac{x^2-3x+5}{(x-1)(x+3)(x-6)}$
9. $\frac{x-5}{x^2-7x+12}$
10. $\frac{2x+3}{x^2-8x+12}$
11. $\frac{4x-7}{x^2-9x+18}$
12. $\frac{3x+8}{6x^2+x-15}$
13. $\frac{5x-1}{15x^2+x-2}$
14. $\frac{1-3x}{5x^2-13x-6}$
15. $\frac{2x+3}{(x^2-x-2)(x+3)}$
16. $\frac{1}{(x-1)(x-3)(x+1)}$
17. $\frac{x^2-x+1}{(x^2-3x+2)(x-3)}$
18. $\frac{x^2+2}{(x^2+4x+3)(x-3)}$
19. $\frac{x+3}{(x-2)(x^2-2x-3)}$
20. $\frac{x-5}{(x-1)(x^2+2x-8)}$
21. $\frac{x^2-2x+2}{(x-2)(x^2+6x+8)}$
22. $\frac{x+1}{(x+4)(x^2-8x+15)}$
23. $\frac{x^2-3}{(x^2-x-6)(x-7)}$
24. $\frac{x^2+x+1}{(x+5)(x^2-5x-14)}$

303. Resolving fractions with repeated linear factors in the denominators.

Example 1. Resolve $\frac{ax+b}{(x-c)^2}$ into partial fractions.

SOLUTION: Let $\frac{ax+b}{(x-c)^2} = \frac{A}{(x-c)^2} + \frac{B}{x-c}$.

Then $ax + b = A + B(x - c)$.

Substituting in order $x = c$ and $x = 0$, we find $A = ac + b$, $B = -ac$. This shows that the given fraction may be thus resolved for all values of a, b, and c. A similar decomposition is possible when the denominator is any integral power of $x - c$. This justifies the general form used in decomposing the fraction (3) on page 332. If in that case $Ax + B$ had been used as a numerator of $(x - 1)^2$, the resulting fraction could again be decomposed as just shown above.

Example 2. Resolve $\dfrac{x^2+5x-2}{(x+3)^2(x-2)}$ into partial fractions.

SOLUTION: Let $\dfrac{x^2+5x-2}{(x+3)^2(x-2)} = \dfrac{A}{(x+3)^2} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$.

Clearing of fractions,

$$x^2+5x-2 = A(x-2) + B(x+3)(x-2) + C(x+3)^2.$$

If $x = -3$, then $-8 = -5A$, and $A = 8/5$.

If $x = 2$, then $12 = 25C$, and $C = 12/25$.

The value of B must now be found by making some other substitution for x. We select that one, not already used, that will make the computation the simplest. Zero nearly always makes this easy, so we shall use that.

Then $-2 = -2A - 6B + 9C$.

Substituting for A and C the values already found we have

$$-2 = -2\cdot\frac{8}{5} - 6B + 9\cdot\frac{12}{25} = -\frac{16}{5} - 6B + \frac{108}{25}$$

and hence $B = \dfrac{13}{25}$.

$A = \dfrac{8}{5}$
$B = \dfrac{13}{25}$
$C = \dfrac{12}{25}$

Therefore,

$$\frac{x^2+5x-2}{(x+3)^2(x-2)} = \frac{8}{5(x+3)^2} + \frac{13}{25(x+3)} + \frac{12}{25(x-2)}.$$

EXERCISES

Resolve the following and prove the first three by adding.

1. $\dfrac{1}{(x+2)(x-1)}$
2. $\dfrac{x^2-x+2}{(x^2+2x+1)(x-3)}$
3. $\dfrac{2x-1}{(x^2-6x+5)(x-1)}$
4. $\dfrac{2x^2-x+2}{(x+3)(2x^2+5x-3)}$
5. $\dfrac{2x-7}{(x-3)(x^2-x-6)}$
6. $\dfrac{2x^2+9x+1}{(3x^2+6x+3)(2x+1)}$
7. $\dfrac{x-3}{(x-1)^2(x+1)}$
8. $\dfrac{x^2+5x-3}{(x+2)^2(x-1)}$
9. $\dfrac{4x^2-1}{(x+2)(x^2+x-2)}$
10. $\dfrac{2x+5}{(2x^2+3x-5)(x-1)}$
11. $\dfrac{7x^2-3}{(x-2)(x^2-4x+4)}$
12. $\dfrac{4x+5}{(x-5)(x^2-3x-10)}$
13. $\dfrac{2x+5}{(x^2+4x+4)(x-1)}$
14. $\dfrac{2x^2+3}{(x^2-2x-15)(x+3)}$

304. ***Polynomials that are identically equal.***—To simplify the work of resolving more complicated fractions into partial fractions, we shall now prove the following fundamental theorem.

If a polynomial $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ *is equal to zero for more than* n *values of* x *then every coefficient* $(a_0, a_1, a_2, \ldots, a_{n-1}, a_n)$ *is equal to zero.*

PROOF: If $x_1, x_2, \ldots, x_n$ are roots of the equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

then by §220 it may be written,

$$a_0(x - x_1)(x - x_2) \ldots (x - x_{n-1})(x - x_n) = 0.$$

If this equation is also satisfied by a value of x, say $x = \bar{x}$ which is different from all of these, then

$$a_0(\bar{x} - x_1)(\bar{x} - x_2) \ldots (\bar{x} - x_{n-1})(\bar{x} - x_n) = 0.$$

Since none of the factors $\bar{x} - x_1$, $\bar{x} - x_2$, etc., is zero, $a_0 = 0$.

Hence the given equation reduces to

$$a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0.$$

By a repetition of the above argument it follows that $a_1 = 0$, then that $a_2 = 0$, and so on to $a_n = 0$.

As a direct corollary of this theorem it follows that if two polynomials of the nth degree,

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

and

$$b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n,$$

are equal for more than n values of x, then $a_0 = b_0$; $a_1 = b_1$; $\ldots$ $a_n = b_n$.

To prove this proposition the two polynomials may be equated, and by transposing, the equation reduced to the form,

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \ldots + a_n - b_n = 0.$$

It follows by the above theorem that $a_0 - b_0 = 0$, $a_1 - b_1 = 0$, $\ldots$ $a_{n-1} - b_{n-1} = 0$, $a_n - b_n = 0$.

The following theorem is now an immediate consequence.

If two polynomials, neither of which is of higher degree than the n*th, are equal for more than* n *values of* x, *they are equal for all values of* x.

305. Resolving fractions having denominators with irreducible factors.—If the equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

with real coefficients has a real root r_1, then $x - r_1$ is a factor of the left member of the equation. If $a + bi$ is a root, then (page 243) $a - bi$ is also a root and $(x - a - bi)(x - a + bi) = x^2 - 2ax + a^2 + b^2$ is a factor of the left member. Factors, such as $x^2 - 2ax + a^2 + b^2$, that do not have real factors are said to be irreducible.

It now follows that any polynomial in x is a product of first degree factors and irreducible second degree factors, all these factors being real.

We have resolved fractions whose denominators are products of first degree real factors. We shall now resolve fractions whose denominators contain second degree irreducible factors.

Suppose we have a fraction whose denominator is of the form $(x^2 + bx + c)(x - d)$, where $x^2 + bx + c$ is irreducible. This fraction may be the sum of two fractions of the form $\dfrac{Ax + B}{x^2 + bx + c}$ and $\dfrac{C}{x - d}$. Hence we must seek to resolve the given fraction into fractions of this form.

Example 1. Resolve $\dfrac{x^2 - 3x + 1}{(x^2 + 3x + 4)(x + 4)}$ into partial fractions.

SOLUTION: $\dfrac{x^2 - 3x + 1}{(x^2 + 3x + 4)(x + 4)} = \dfrac{Ax + B}{x^2 + 3x + 4} + \dfrac{C}{x + 4}$

or $x^2 - 3x + 1 = (Ax + B)(x + 4) + C(x^2 + 3x + 4)$

or $x^2 - 3x + 1 = Ax^2 + 4Ax + Bx + 4B + Cx^2 + 3Cx + 4C$

or $x^2(A + C - 1) + x(4A + B + 3C + 3) + 4B + 4C - 1 = 0.$

Hence by §304, we have the equations at the right. Solving these gives

$$\begin{aligned} A + C - 1 &= 0 \\ 4A + B + 3C + 3 &= 0 \\ 4B + 4C - 1 &= 0 \end{aligned}$$

$A = -21/8$, $B = -27/8$, $C = 29/8$.

Hence $\dfrac{x^2 - 3x + 1}{(x^2 + 3x + 4)(x + 4)} = -\dfrac{21x + 27}{8(x^2 + 3x + 4)} + \dfrac{29}{8(x + 4)}$

Check this solution by adding the fractions in the right member of this equation.

Example 2. Resolve $\dfrac{x^2+8x-7}{(x^2+2x+6)(x^2-5x-9)}$ into partial fractions.

SOLUTION:

$$\frac{x^2+8x-7}{(x^2+2x+6)(x^2-5x-9)} = \frac{Ax+B}{x^2+2x+6} + \frac{Cx+D}{x^2-5x-9}$$

or $x^2+8x-7 = (Ax+B)(x^2-5x-9) + (Cx+D)(x^2+2x+6)$.

Multiplying and collecting terms, we have $(A+C)x^3 + (-5A+B+2C+D-1)x^2 + (-9A-5B+6C+2D-8)x - 9B + 6D + 7 = 0$.

Hence we have the equations at the right.

$$\begin{aligned} A + C &= 0 \\ -5A + B + 2C + D - 1 &= 0 \\ -9A - 5B + 6C + 2D - 8 &= 0 \\ -9B + 6D + 7 &= 0 \end{aligned}$$

Solving these equations gives

$$A = -181/309,\ B = -239/309,\ C = 181/309,\ D = -719/309.$$

Hence

$$\frac{x^2+8x-7}{(x^2+2x+6)(x^2-5x-9)} = -\frac{181x+239}{309(x^2+2x+6)} + \frac{181x-719}{309(x^2-5x-9)}$$

If a denominator has a factor like x^2+4x-4, which can be factored into the irrational (but real) factors $(x+2+2\sqrt{2})(x+2-2\sqrt{2})$, the work of decomposing the fraction is done exactly as in §302, but is more laborious.

MISCELLANEOUS EXERCISES

Resolve each of the following into partial fractions.

1. $\dfrac{5x-3}{(x^2+8x+6)(x+1)}$
2. $\dfrac{2x+7}{(x^2-3x+4)(x^2-1)}$
3. $\dfrac{5x-3}{(x^2+8x+6)(x^2+1)}$
4. $\dfrac{x^3-2x+4}{(4x^2+1)(x^2+6)}$
5. $\dfrac{x^2-13}{(3x^2+2)(x^2+x+1)}$
6. $\dfrac{3x-2}{(x^2+3)(x^2+4x+5)}$
7. $\dfrac{4}{(x^2-3x+8)(x^2+5)}$
8. $\dfrac{2x^2+3}{(x^2+2)(x^2-3x-5)}$
9. $\dfrac{2x^2-3}{(x^2+x+6)(x^2-3x+4)}$
10. $\dfrac{x^3-4x^2-1}{(x^2-5x+3)(2x^2-x+3)}$
11. $\dfrac{x^2+2x-3}{(4x^2-4x+3)(2x^2-3x-4)}$
12. $\dfrac{x^2-8}{(x^2+x+12)(x-2)}$
13. $\dfrac{x^3-1}{(x^2-5x+6)(x^2+4x+4)}$
14. $\dfrac{x^2-x-1}{(x^2+2x+4)(x^2+3)}$
15. $\dfrac{x^3+2x-1}{(x^2+7x+12)(x^2+x+1)}$
16. $\dfrac{3x^2-2x+2}{(x^2+x+4)(x^2-x+4)}$

306. Permissible substitutions; identities.—We shall now restate in a somewhat different form the substance of §§81, 82. In a polynomial in x any number whatever may be substituted for the variable x. In a fraction such as $\frac{x+1}{x-1}$ any number except $x = 1$ may be substituted. The substitution $x = 1$ is not permitted because that reduces the denominator to zero. By §20 we know that division by zero is impossible under all circumstances. Any substitution in an algebraic expression which does not reduce a denominator (divisor) to zero is said to be permissible. An equation that is satisfied by every permissible substitution is an identity.

Thus $(x+3)^2 = x^2 + 6x + 9$ and $\frac{x^2-4}{x-2} = x + 2$ are identities, though the second of these is not satisfied by $x = 2$, since that is not a permissible substitution in $\frac{x^2-4}{x-2}$.

Theorem 1. If the members of an identity are increased by, decreased by, multiplied by, or divided by the same expression, the result is an identity.

This theorem is an immediate consequence of the uniqueness of the four fundamental operations of arithmetic. (See §§83, 84.)

Any one of these steps may introduce or take away numbers that will satisfy the equation. The theorem nevertheless holds. Thus clearing the above equation of fractions reduces it to $2 = -x + 1 + x + 1$, both of which are identities, though the latter is satisfied by $x = 1$, $x = -1$, while the given equation is not.

Theorem 2. If one member of an equality can be reduced to exactly the same form as the other by carrying out the indicated operations, then the equality is an identity.

As an illustration consider the equality at the right. By carrying out the indicated operation, the right member reduces to the same as the left member. Hence this equality is an identity. Note that the substitutions $x = 1$, $x = -1$ are not permitted. For every other value of x the two members represent the same number.

$$\frac{1}{x^2-1} = -\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$$

However, clearing this equation of fractions results in the equation $2 = -(x-1) + (x+1)$, in which all substitutions for x are permissible.

307. Theorems showing reducibility of fractions.—Theorem 1. Let $\frac{N(x)}{D(x)}$ *be a proper rational fraction in its lowest terms, such that*

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(x-a)^p \cdot D_1(x)}$$

where $x - a$ *is not a factor of* $D_1(x)$. *Then there is a number* A, *different from zero, and a polynomial* $N_1(x)$ *of degree lower than* $(x-a)^{p-1} \cdot D_1(x)$ *such that*

$$\frac{N(x)}{D(x)} = \frac{A}{(x-a)^p} + \frac{N_1(x)}{(x-a)^{p-1}D_1(x)} \tag{1}$$

is an identity in x.

PROOF: Clearing (1) of fractions

$$N(x) = A \cdot D_1(x) + (x-a)N_1(x). \tag{2}$$

In this equality $N(x)$ and $D_1(x)$ are given and we are to find values of A and the coefficients in $N_1(x)$ that will make the equality an identity. We must also show that $N_1(x)$ is of lower degree than $(x-a)^{p-1}D_1(x)$.

If (2) is to be an identity, it must be satisfied by $x = a$. This gives $N(a) = A \cdot D_1(a) + (a-a)N_1(a)$, or $A = \frac{N(a)}{D_1(a)}$, where $D_1(a) \neq 0$.

Since $N(x)$ and $D_1(x)$ are completely determined, it follows that if the terms of (2) are transposed into the first member of the equation, we may equate the coefficients of each power of x and find the coefficients in the polynomial $(x-a)N_1(x)$.

Clearly $(x-a)N_1(x)$ cannot be of higher degree than the highest degree in $N(x)$ or $AD_1(x)$, since the coefficients of such a term in $(x-a)N_1(x)$ must be zero because the term does not occur either in $N(x)$ or in $AD_1(x)$. Hence $(x-a)N_1(x)$ is of lower degree than $D(x) = (x-a)^pD_1(x)$, and $N_1(x)$ is therefore of lower degree than $(x-a)^{p-1}D_1(x)$.

This process may be repeated until all factors $x - a$ are removed. That is,

$$\frac{N(x)}{(x-a)^pD_1(x)} = \frac{A}{(x-a)^p} + \frac{B}{(x-a)^{p-1}} + \cdots + \frac{N_p(x)}{D_1(x)}.$$

While A is necessarily different from zero, any one or more of the other constant numerators may be zero. Thus B may be zero because

$x - a$ may be a factor of $N_1(x)$ so that the fraction $\dfrac{N_1(x)}{(x-a)^{p-1}D_1(x)}$ may reduce to $\dfrac{N_2(x)}{(x-a)^{p-2}D_1(x)}$, or possibly to lower terms.

This is shown in the following example.

Let $$\frac{2x^3 - 3x^2 + x + 3}{(x-1)^3(x+2)} = \frac{A}{(x-1)^3} + \frac{N(x)}{(x-1)^2(x+2)}.$$

Proceeding as in the theorem, $A = 1$ and $N_1(x) = 2x^2 - x - 1$. But $x - 1$ is a factor of $2x^2 - x - 1 = (x-1)(2x+1)$.

Hence $$\frac{2x^3 - 3x^2 + x + 3}{(x-1)^3(x+2)} = \frac{1}{(x-1)^3} + \frac{2x+1}{(x-1)(x+2)},$$

and therefore in this case there can be no term of the type $\dfrac{B}{(x-1)^2}$.

Clearly any other linear factor can be removed in this manner and we are left with a fraction whose denominator has irreducible quadratic factors and no others.

Theorem 2. Let $\dfrac{N(x)}{D(x)}$ *be a proper fraction in its lowest terms such that*

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(ax^2 + bx + c)^q D_1(x)},$$

where $ax^2 + bx + c$ *is not a factor of* $D_1(x)$.
Then

$$\frac{N(x)}{D(x)} = \frac{Ax + B}{(ax^2 + bx + c)^q} + \frac{N_1(x)}{(ax^2 + bx + c)^{q-1}D_1(x)} \tag{1}$$

where A *or* B *may be zero, but not both, and where* $N_1(x)$ *is a polynomial of degree lower than* $(ax^2 + bx + c)^{q-1}D_1(x)$.

PROOF: Let n be the degree of $D(x)$.
Write $N_1(x) = a_0x^{n-3} + a_1x^{n-4} + \cdots + a_{n-3}$ and clear (1) of fractions.
Then

$$N(x) = (Ax + B)D_1(x) + N_1(x)(ax^2 + bx + c). \tag{2}$$

There are n constants, $A, B, a_0, a_1, \ldots, a_{n-3}$ to be determined and equation (2) is of degree $n - 1$. Hence, equating coefficients of like powers to zero we have n equations, which are exactly enough to determine these constants.

Either A or B may be zero, but not both, since in that case $\dfrac{N(x)}{(ax^2+bx+c)^q D_1(x)}$ would be reducible to $\dfrac{N_1(x)}{(ax^2+bx+c)^{q-1}D_1(x)}$ and the given fraction would not be in its lowest terms.

As in the case of theorem 1, $N_1(x)$ may contain ax^2+bx+c as a factor and hence the resulting fraction may not be in its lowest terms. This is shown in the following example.

Let

$$\frac{x^4+2x^3+3x^2+3x-2}{(x^2+x+1)^2(x-3)} = \frac{Ax+B}{(x^2+x+1)^2} + \frac{N_1(x)}{(x^2+x+1)(x-3)}, \qquad (3)$$

and let $N_1(x) = Cx^2+Dx+E$.

Clearing (3) of fractions,

$$\begin{aligned} x^4+2x^3+3x^2+3x-2 &= (Ax+B)(x-3)+(Cx^2+Dx+E)(x^2+x+1) \\ &= Cx^4+(C+D)x^3+(A+C+D+E)x^2 \\ &\quad +(B-3A+D+E)x+E-3B. \end{aligned}$$

Equating coefficients and solving gives

$$A=0,\ B=1,\ C=1,\ D=1,\ E=1.$$

Hence
$$\frac{x^4+2x^3+3x^2+3x-2}{(x^2+x+1)^2(x-3)} = \frac{1}{(x^2+x+1)^2} + \frac{x^2+x+1}{(x^2+x+1)(x-3)} = \frac{1}{(x^2+x+1)^2} + \frac{1}{x-3}$$

By repeated use of theorems 1 and 2, any rational fraction may be broken up into partial fractions of the types $\dfrac{A}{(x-a)^p}$ and $\dfrac{Ax+B}{(ax^2+bx+c)^q}$, in the second of which A or B may be zero.

The possibility of thus decomposing a fraction depends on the possibility of finding all factors of the denominator which are of the types $x-a$ and ax^2+bx+c. In practice this may be quite impossible. But the factors once found, the fraction may be decomposed.

CHAPTER 25:

DETERMINANTS

In Chapter 8 we used determinants of the second and third order in solving simultaneous linear equations. We shall begin the present chapter by restudying these same determinants, developing certain definitions and theorems that hold also for determinants of higher orders. The general proofs of these theorems will be found in the later part of the chapter.

308. Definition of second and third order determinants.—It should be clear that, from the purely logical point of view, definitions are arbitrary. We may adopt any definitions that we wish. But once they are adopted, we must adhere to them as long as we continue to use the terms that we have defined, unless, indeed, we state explicitly that we are adopting different definitions. However, from the practical point of view there are two considerations that we need to take into account: (1) If a term that we are defining is in general use among those dealing with our subject, then we must use it in a sense that will seem natural to others. That is, we must adopt the definition that is in current use. (2) A definition of a mathematical term should be such that it will be conducive to simplicity and smoothness in the theory we are developing.

With these ideas in mind we give the following definitions of the second and third order determinants.[1]

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

For a scheme for reading these determinants, see page 345.

[1] Note the way letters and subscripts are used in these determinants. Thus the letters a, b, c indicate columns, while the subscripts $_1$, $_2$, $_3$ indicate rows.

We note that each determinant consists of a square array of numbers. The second order determinant is a square array of 4 numbers, written in two columns (and two rows). The third order determinant consists of 9 numbers written in 3 columns (and 3 rows). We shall see later that an nth order determinant consists of n^2 numbers written in n columns (and n rows). (See page 359.)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The polynomials at the right in each of these equalities are called the expansions of the determinants. The determinants are arbitrarily chosen symbols, which by definition have the meaning given by this expansion.

309. ***Solving linear equations by determinants.***—In Chapter 8 we found that on solving the two linear equations at the right,

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

and that the numerators are the expansions of the two determinants $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, while the denominator in each fraction is the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. On solving the three linear equations at the right we found the solutions

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

which are quotients of third order determinants. While these would be extremely difficult to remember if we did not take notice of certain orderliness in the arrangement of the letters and the subscripts, the determinant forms are very easily remembered. In fact it was in connection with solving linear equations that the determinants were discovered. (See page 412 in the Historical Sketch.)

Written in the determinant form these solutions are

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The three denominators are identical. The elements as they stand may be regarded as formed by "lifting" the coefficients in the left members of the equations exactly in the positions in which they stand. This determinant is called the determinant of the system, and is denoted by $\triangle$.

The numerator in the value of x is obtained by replacing the a's (the coefficients of x) in $\triangle$ by the d's of the equations, the numerator in the value of y is obtained by replacing the b's (the coefficients of y) by the d's, and the numerator in the value of z is obtained by replacing the c's by the d's.

The values of x, y, z are now written:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\triangle}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\triangle}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\triangle}.$$

EXERCISE

Using the scheme at the right (see Chapter 8) expand the determinants in the numerators given above and then show that they give the solution shown on page 344.

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

EXERCISES

Expand the following determinants.

1. $\begin{vmatrix} 4 & 7 \\ 3 & 5 \end{vmatrix}$
2. $\begin{vmatrix} 5x & b \\ 2a & x \end{vmatrix}$
3. $\begin{vmatrix} 2a & 4b \\ 3a & b \end{vmatrix}$
4. $\begin{vmatrix} 6ab & 2c \\ 4a^2 & c^2 \end{vmatrix}$
5. $\begin{vmatrix} 1 & 3 & 5 \\ a & b & x \\ b & a & x \end{vmatrix}$
6. $\begin{vmatrix} 2 & 1 & 2 \\ 3 & 0 & 2 \\ 4 & 6 & 5 \end{vmatrix}$
7. $\begin{vmatrix} 3 & 2 & y \\ x & 3 & 1 \\ y & x & 0 \end{vmatrix}$
8. $\begin{vmatrix} 8 & a & a \\ 2 & b & 2 \\ 4 & 2 & b \end{vmatrix}$

Using determinants, solve the following equations.

9. $2x + y - z = 4$
$x - y + 3z = 2$
$-x + 2y - 2z = 1$

10. $x + y + z = 10$
$2x - 3y + 2z = 8$
$3x + 2y - 6z = 12$

11. $5a - 3b + c = 12$
$2a + 3b - 3c = 10$
$a + 2b + 3c = 16$

12. $x - 2y + z - 16 = 0$
$5x - 3y - z + 4 = 0$
$2x + 3y + 2z - 20 = 0$

13. $2p + 3q + 4r = 20$
$3p - q + 2r - 9 = 0$
$p - 4q + 3r - 16 = 0$

14. $7x - 4y + z = 6$
$3x + 2y + z = 11$
$2x - 3y + 4z = 18$

15. $3x - 2y + z - 9 = 0$
$-2x + y - 4z + 2 = 0$
$7x - 5y - 3z + 15 = 0$

16. $7r - 3s + 9t = 11$
$-6r + 9s - 2t = 5$
$2r + 7s + 8t = 24$

17. $9a - 8b + 2c - 27 = 0$
$5b + 7a - 9c - 18 = 0$
$3c + 8b - 4a - 12 = 0$

310. Theorems stating properties of determinants.—We shall now state a series of theorems on determinants, establishing each of them for second and third order determinants by actually expanding the determinants. General proofs of these theorems are given on pages 360–369. The present purpose is to acquire clear ideas of the actual meaning and use of these theorems. We should bear in mind that they apply, in many cases with obvious modifications or extensions, to determinants of any order.

Theorem 1. The value of a determinant is not changed by changing its rows into columns and columns into rows.

To prove this theorem we expand each determinant at the right.

$$(1)\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (2)\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(1) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$
$$(2) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

To prove this theorem for second order determinants, we compare

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

EXERCISE

Check theorem 1 by expanding the determinants.

$$\begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & -9 \\ 3 & 5 & 6 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 1 & -9 & 6 \end{vmatrix}. \quad \text{Also} \quad \begin{vmatrix} 5 & 2 & 4 \\ -3 & 6 & 0 \\ -2 & 3 & 8 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 5 & -3 & -2 \\ 2 & 6 & 3 \\ 4 & 0 & 8 \end{vmatrix}.$$

Theorem 2. If two columns (or two rows) of a determinant are interchanged its sign is changed but not its absolute value.

$$\overset{(1)}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad \overset{(2)}{\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}} \qquad \overset{(3)}{\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}} \qquad \overset{(4)}{\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}}$$

If a's and b's in (1) are interchanged, the result is (2); if a's and c's in (1) are interchanged, the result is (3); and if b's and c's in (1) are interchanged, the result is (4).

EXERCISES

1. Expand determinants (1) and (2) above and compare the results. Does the result verify theorem 2 above?

2. Expand determinants (1) and (4) above and show that the result verifies the same theorem.

3. Expand the two determinants at the right. What theorem is verified by the results?

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expand each of the following and then interchange one pair of columns or rows and check theorem 2 by expanding again.

4. $\begin{vmatrix} 3 & 1 & 7 \\ 2 & -1 & 3 \\ 4 & 2 & 6 \end{vmatrix}$ **5.** $\begin{vmatrix} a & 3 & 4 \\ b & 1 & 2 \\ c & 3 & 1 \end{vmatrix}$ **6.** $\begin{vmatrix} x & 1 & 4 \\ 2x & 2 & 2 \\ 3x & 3 & 1 \end{vmatrix}$

7. $\begin{vmatrix} 6 & -a & b \\ 3 & -b & c \\ 4 & 3 & 1 \end{vmatrix}$ **8.** $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ **9.** $\begin{vmatrix} k & 3 & 6 \\ 4k & 1 & 3 \\ 8k & 2 & 9 \end{vmatrix}$

10. $\begin{vmatrix} 8 & -1 & -8 \\ 6 & 4 & 3 \\ x & y & z \end{vmatrix}$ **11.** $\begin{vmatrix} m & -n & p \\ q & r & s \\ 3 & 4 & 7 \end{vmatrix}$ **12.** $\begin{vmatrix} 2a & b & 3 \\ b & c & a \\ 3c & a & b \end{vmatrix}$

13. $\begin{vmatrix} 7 & -2 & 7 \\ 2 & 9 & 3 \\ a & 4 & 7 \end{vmatrix}$ **14.** $\begin{vmatrix} 8 & x & 3 \\ 2 & 2x & 7 \\ x & 5 & 9 \end{vmatrix}$ **15.** $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

16. $\begin{vmatrix} 1 & 2 & a \\ x & 3 & b \\ y & 4 & c \end{vmatrix}$ **17.** $\begin{vmatrix} 7 & 3 & b \\ 4 & b & c \\ 6 & b & c^2 \end{vmatrix}$ **18.** $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

19. $\begin{vmatrix} r & s & t \\ 3 & 1 & 7 \\ 2 & 3 & 6 \end{vmatrix}$ **20.** $\begin{vmatrix} p & q & 6 \\ r & p & 4 \\ p & 1 & 9 \end{vmatrix}$ **21.** $\begin{vmatrix} 8 & r & x \\ 9 & 3 & r \\ 7 & x & 8 \end{vmatrix}$

Theorem 3. If all the elements in one column (or row) in a determinant are multiplied by the same number, then the value of the determinant is multiplied by that number.

PROOF: If the elements in the first column of our standard determinant (at the right) are multiplied by k and the resulting determinant is expanded, then the a's in the expansion

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

are replaced by ka_1, ka_2, ka_3, and hence every term is multiplied by k. This proves the theorem.

If in the preceding $\frac{1}{k}$ is used instead of k we have the following.

Theorem 4. If all the elements in one column (or row) in a determinant are divided by the same number, then the value of the determinant is divided by that number.

It follows that a factor common to all the elements of a column (or row) may be taken out before expanding a determinant. Thus

$$\begin{vmatrix} 4 & 3 & 5 \\ 6 & 9 & 10 \\ 8 & 12 & 25 \end{vmatrix} = 2 \times 3 \times 5 \times \begin{vmatrix} 2 & 1 & 1 \\ 3 & 3 & 2 \\ 4 & 4 & 5 \end{vmatrix}.$$

In this case the factor 2 is removed from the first column, the factor 3 from the second column, and the factor 5 from the third column.

EXERCISES

1. Verify theorem 3 by multiplying different columns and rows of the general third order determined by k, to see whether each of these multiplies the expansion by k.

2. Expand the determinant at the left above and also the determinant resulting after the factors 2, 3, and 5 have been taken out. Thus verify Theorem 4.

Find the values of the following by first taking out factors.

3. $\begin{vmatrix} 2 & 1 & 6 \\ 4 & 3 & 3 \\ 6 & 9 & 9 \end{vmatrix}$ 4. $\begin{vmatrix} 4 & 8 & 16 \\ 0 & 3 & 6 \\ 5 & 15 & 10 \end{vmatrix}$ 5. $\begin{vmatrix} a & a^2 & a^3 \\ b^3 & b & b^2 \\ x^2 & 2x & 3x \end{vmatrix}$ 6. $\begin{vmatrix} 6a & 3 & 9b \\ a^2 & 9 & 4b \\ a & 12 & 6b^2 \end{vmatrix}$

7. $\begin{vmatrix} 8 & 4 & 6 \\ 3 & 9 & 12 \\ 4 & 6 & 10 \end{vmatrix}$ 8. $\begin{vmatrix} r & s & t \\ r & s^2 & t^2 \\ r^2 & s & t^2 \end{vmatrix}$ 9. $\begin{vmatrix} 1 & 2 & 3 \\ p^2 & p^3 & p^4 \\ r^2 & r & r^3 \end{vmatrix}$ 10. $\begin{vmatrix} ab & b^2 & 5 \\ ac & c^2 & 10 \\ ad & d^2 & 15 \end{vmatrix}$

Theorem 5. If two columns (or rows) in a determinant are identical, the value of the determinant is zero.

PROOF: If the two identical columns (rows) are interchanged, then the sign of the determinant is changed (Theorem 1). But this change leads to a determinant that is identical with the given determinant. Hence if D represents the value of the determinant, then $D = -D$ and therefore, $D = 0$.

311. Minors in a determinant.—As we shall see presently, the idea of minors in a determinant is of importance in finding its value. Consider a third order determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} -a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The minor of a_1 in this determinant is the second order determinant that remains if we strike out the column and row in which a_1 appears. That is, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is the minor of a_1. This minor is usually represented by A_1.

In general, the minor of any element of a determinant is the determinant remaining after the column and row in which the element appears are stricken out. The minor of a_2 is represented by A_2, the minor of a_3 is represented by A_3, the minor of b_1 by B_1, and so on. Thus in the above determinant we have:

$$A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \quad C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

EXERCISES

1. Using the above definition of minors, verify that the minors just given are correct.

2. In the determinant at the right give the minor of each of the nine elements in the determinant.

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

3. Expand the determinant at the right, thus verifying Theorem 5 above. Also write a determinant with identical rows and expand it, thus again verifying the same theorem.

$$\begin{vmatrix} 3 & 3 & 2 \\ 4 & 4 & 6 \\ 1 & 1 & 3 \end{vmatrix}$$

4. Evaluate the determinant at the right. Try to find the value of this determinant without actually expanding it. What theorem can you use?

$$\begin{vmatrix} 3 & 6 & 9 \\ 1 & 7 & 3 \\ -2 & 3 & -6 \end{vmatrix}$$

312. Expanding determinants in terms of minors.—By examining the expansion

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

we see at once that a_1 appears in the first and the last terms, a_2 appears in the second and fifth terms, a_3 appears in the third and fourth terms. Thus the expansion may be written

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

But $\quad A_1 = b_2c_3 - b_3c_2,\ A_2 = b_1c_3 - b_3c_1,\ A_3 = b_1c_2 - b_2c_1.$

Hence the expansion is

$$a_1A_1 - a_2A_2 + a_3A_3$$

This expansion is therefore in terms of the elements of the first column and their minors. In this expansion the signs alternate, beginning with the plus sign.

The determinant may be evaluated in terms of the elements of any column (or row) and their minors. The signs of the terms are shown in the following, D being used to represent the determinant.

$$\begin{aligned} D &= a_1A_1 - a_2A_2 + a_3A_3 = -b_1B_1 + b_2B_2 - b_3B_3 = c_1C_1 - c_2C_2 + c_3C_3 \\ &= a_1A_1 - b_1B_1 + c_1C_1 = -a_2A_2 + b_2B_2 - c_2C_2 = a_3A_3 - b_3B_3 + c_3C_3. \end{aligned}$$

In each case the signs alternate. Evaluating in terms of the elements of the first column and their minors, we begin with the plus sign; evaluating in terms of the elements of the second column and their minors, we begin with the minus sign; and so on.

EXERCISES

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

1. Expand the determinant at the right in terms of the elements of the first column and their minors. Then find the values of the minors and then the value of the determinant.

2. Expand the same determinant in terms of the elements of the second column and their minors. Then expand it in terms of the elements of the third column.

3. Expand the same determinant in terms of the elements of each of its rows. Check that all these expansions lead to the same value of the determinant.

313. Addition of determinants.—We shall begin by decomposing a determinant into two determinants. Let the elements of the first column of a determinant be the sum of two numbers as in the determinant at the right. Denoting the minors of $a_1 + a_1'$, $a_2 + a_2'$, $a_3 + a_3'$ by A_1, A_2, A_3 and expanding in terms of the elements of the first column and their minors, we have

$$\begin{vmatrix} a_1 + a_1' & b_1 & c_1 \\ a_2 + a_2' & b_2 & c_2 \\ a_3 + a_3' & b_3 & c_3 \end{vmatrix}$$

$$\begin{aligned} &(a_1 + a_1')A_1 - (a_2 + a_2')A_2 + (a_3 + a_3')A_3 \\ &= a_1A_1 - a_2A_2 + a_3A_3 + a_1'A_1 - a_2'A_2 + a_3'A_3 \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

It follows that if two determinants are identical except in one column (or row), then the determinants may be added by adding the elements in the unlike columns (or rows) leaving the identical columns (rows) unchanged.

Thus $$\begin{vmatrix} 5 & 3 & 2 \\ 1 & 9 & -6 \\ 7 & 2 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 3 & 2 \\ 3 & 9 & -6 \\ 9 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 9 & 3 & 2 \\ 4 & 9 & -6 \\ 16 & 2 & 4 \end{vmatrix}$$

and $$\begin{vmatrix} a & b & c \\ 2 & -4 & 6 \\ 8 & 5 & 3 \end{vmatrix} + \begin{vmatrix} a & b & c \\ -8 & 2 & -4 \\ 8 & 5 & 3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ -6 & -2 & 2 \\ 8 & 5 & 3 \end{vmatrix}$$

Clearly the two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1' & kb_1 & c_1 \\ a_2' & kb_2 & c_2 \\ a_3' & kb_3 & c_3 \end{vmatrix}$$

may be added in this way by writing the second determinant so that k is multiplied into the first column instead of the second.

That is $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & kb_1 & c_1 \\ a_2' & kb_2 & c_2 \\ a_3' & kb_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_1' & b_1 & c_1 \\ a_2 + ka_2' & b_2 & c_2 \\ a_3 + ka_3' & b_3 & c_3 \end{vmatrix}$$

EXERCISE

Find the sum of the two determinants at the right (a) by adding the determinants and expanding the result, (b) by expanding first and then adding.

$$\begin{vmatrix} 3 & 2 & 4 \\ 4 & -1 & 7 \\ 7 & -3 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 2 & 4 \\ 7 & -1 & 7 \\ 3 & -3 & 2 \end{vmatrix}$$

314. Expanding determinants by adding or subtracting multiples of columns or rows.—The principle for adding determinants just developed may be used effectively in finding the numerical value of determinants. The following theorem is now easily established.

Theorem. If any multiples of the elements of one column (row) are added to the elements of another column (row), the determinant remains unchanged in value.

PROOF. Consider the determinants at the right. By theorem 4, §310, the second determinant at the right is equal to zero. Hence adding this determinant to the first leaves the sum unchanged. That is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In practice we may add equimultiples of one column (row) to the elements of another column (row), and then different equimultiples of the same column to those of another column. An example will show how this may be done to achieve our purpose.

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ 3 & -6 & -12 \end{vmatrix} = 1\begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 1 \cdot 3 \cdot 6\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

STEP 1. Subtract 4 times the elements of the first column from those of the second.

STEP 2. Subtract 7 times the elements of the first column from those of the third column.

These steps give the second determinant above. The purpose in selecting the multiples 4 and 7 of the first column is to leave the elements 0 in the first row as shown in the second determinant. Note that the first column is left unchanged. This determinant is now expanded in terms of the elements of the first row and their minors. We then have

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ 3 & -6 & -12 \end{vmatrix} = 1\begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & -6 \\ 3 & -9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & -3 \\ 3 & -6 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix}$$

This second order determinant is now evaluated by taking out the factors -3 and -6 from the two columns, leaving

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0.$$

In evaluating this determinant we might equally well have proceeded as follows: From the elements of the second row subtract twice the elements of the first, and from the elements of the third row subtract 3 times the elements of the first. Then we have

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 0.$$

Compare the expansion by this method with the more direct but less expeditious method used on page 350 for the same determinant.

A point to be noticed is that in this process one column or row must be left as it is. With this restriction we may add or subtract equimultiples as we please.

Example 1. Expand the determinant at the left below.

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{vmatrix} = \tfrac{1}{4}\begin{vmatrix} 2 & 3 & 5 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{vmatrix} = \tfrac{1}{4}\begin{vmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & -1 & -7 \end{vmatrix} = \tfrac{1}{4}\cdot 2\begin{vmatrix} 1 & -1 \\ -1 & -7 \end{vmatrix} = \tfrac{1}{2}(-7 - 1) = -4.$$

In this determinant we cannot multiply by an integer any column or row and subtract so as to get all elements zero except one. We now decide to keep the first row intact; then multiply the second and third rows by 2 and hence put the factor ¼ before the determinant. The determinant is now expanded as shown. Explain each step.

Expand the above determinant by starting as at the right. How does this compare in complexity with the above?

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{vmatrix} = \frac{1}{9}\begin{vmatrix} 2 & 3 & 5 \\ 9 & 15 & 21 \\ 15 & 21 & 27 \end{vmatrix}$$

Example 2. Expand the determinant at the left below

$$\begin{vmatrix} 3 & -4 & 7 \\ 4 & 8 & -3 \\ -3 & 16 & -2 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 & 7 \\ 4 & 2 & -3 \\ -3 & 4 & -2 \end{vmatrix} = 4\begin{vmatrix} 0 & -1 & 0 \\ 10 & 2 & 11 \\ 9 & 4 & 26 \end{vmatrix} = 4\begin{vmatrix} 10 & 11 \\ 9 & 26 \end{vmatrix} = 4(260 - 9\cdot 11) = 644.$$

In this determinant we take out the factor 4 in the second column. Then we add 3 times this column to the first column and 7 times this column to the third, and expand according to minors of the first row.

The second determinant above may also be expanded as follows.

$$4\begin{vmatrix} 3 & -1 & 7 \\ 4 & 2 & -3 \\ -3 & 4 & -2 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 & 7 \\ 10 & 0 & 11 \\ 9 & 0 & 26 \end{vmatrix} = 4\begin{vmatrix} 10 & 11 \\ 9 & 26 \end{vmatrix} = 4(260 - 99) = 4\cdot 161 = 644$$

The row or column to be retained unchanged must be a matter of judgment. If in the first determinant at the top of the page we tried to keep the last column, for example, fixed we should have considerable trouble.

Example 3. Expand the first determinant below.

$$\begin{vmatrix} 11 & 8 & 3 \\ 3 & -2 & 0 \\ 5 & 7 & 5 \end{vmatrix} = \tfrac{1}{3}\begin{vmatrix} 11 & 8 & 3 \\ 3 & -2 & 0 \\ 15 & 21 & 15 \end{vmatrix} = \tfrac{1}{3}\begin{vmatrix} 11 & 8 & 3 \\ 3 & -2 & 0 \\ -40 & -19 & 0 \end{vmatrix} = \tfrac{1}{3}\times 3\begin{vmatrix} 3 & -2 \\ -40 & -19 \end{vmatrix}$$
$$= -57 - 80 = -137$$

Since the third column contains one zero, we decide to make one more figure zero in that column. Multiply the last row by 3 and then subtract 3 times the first row. The remaining steps shown above are now easily understood.

315. Rule for expanding determinants.

1. Take out any factor common to the elements of any column (or row).

2. Add multiples of any column (row) to the elements of the other columns (rows) so as to make all elements except one in some row (column) equal to zero. Instead of adding, we may of course subtract.

3. Expand the determinant in terms of the elements and minors of the row (column) containing the zeros.

316. Cofactors of the elements of a determinant.—The expression cofactor of an element of a determinant is used with exactly the same meaning as minor of an element with the exception that the signs are not the same.

In the determinant at the right the minors of the terms a_1, a_2, a_3 are $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, while the cofactors of a_1, a_2, a_3 are A_1, $-A_2$, A_3 respectively.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The expansion of the determinant in terms of minors is $a_1A_1 - a_2A_2 + a_3A_3$, while in terms of cofactors it is $a_1A_1 + a_2A_2 + a_3A_3$.

The sign of the cofactor of a certain element in a determinant is determined according to the following rule.

Start with the first element in the first row and count elements along the first row to the column in which the element is found and then down this column to the given element. If the count is an odd number, the cofactor of the element is positive, and if the count is even, the cofactor is negative. Thus in the given determinant the cofactor of a_1 is positive, of b_1 it is negative, of c_1 it is positive, of c_2 it is negative, and so on.

In this book we shall use minors instead of cofactors.

The process we have described will reduce the determinant to the next lower order.

In this process care must be taken to keep one column (row) unchanged. Skill in selecting the row (column) in which the zeros are to appear counts greatly in facilitating the work of evaluating determinants.

EXERCISES

1. Give the first step in evaluating each of the following determinants.

(A) $\begin{vmatrix} 3 & -4 & 6 \\ 5 & 2 & -3 \\ 6 & 7 & -9 \end{vmatrix}$ (B) $\begin{vmatrix} -2 & 5 & 6 \\ 4 & 3 & 2 \\ 10 & -5 & 8 \end{vmatrix}$ (C) $\begin{vmatrix} 7 & 14 & -21 \\ 3 & 9 & 7 \\ 2 & 1 & 4 \end{vmatrix}$ (D) $\begin{vmatrix} 4 & 3 & 2 \\ 2 & 6 & -10 \\ 6 & -12 & 18 \end{vmatrix}$

2. Give the first step in evaluating each of the following determinants.

(A) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & -3 \\ 6 & 2 & -5 \end{vmatrix}$ (B) $\begin{vmatrix} 8 & 3 & 1 \\ 6 & 0 & 2 \\ 5 & 4 & 3 \end{vmatrix}$ (C) $\begin{vmatrix} 7 & 0 & 0 \\ 4 & 9 & 6 \\ 7 & 8 & 10 \end{vmatrix}$ (D) $\begin{vmatrix} 3 & 5 & 2 \\ 4 & 7 & 5 \\ 2 & 9 & 6 \end{vmatrix}$

Evaluate the following determinants.

3. $\begin{vmatrix} 6 & 1 & 3 \\ 2 & 5 & 7 \\ 4 & 3 & 5 \end{vmatrix}$ 4. $\begin{vmatrix} 1 & -5 & 4 \\ 6 & 8 & 4 \\ 7 & 3 & 6 \end{vmatrix}$ 5. $\begin{vmatrix} 9 & 5 & 8 \\ 8 & 2 & 7 \\ 6 & 1 & 4 \end{vmatrix}$ 6. $\begin{vmatrix} 3 & 2 & 9 \\ 8 & 7 & 2 \\ 9 & 9 & 1 \end{vmatrix}$

7. $\begin{vmatrix} 8 & 4 & 3 \\ -9 & 7 & 5 \\ 4 & 2 & 8 \end{vmatrix}$ 8. $\begin{vmatrix} 8 & 3 & 4 \\ 2 & 4 & 6 \\ 1 & 9 & 4 \end{vmatrix}$ 9. $\begin{vmatrix} 2 & 0 & 5 \\ 3 & 6 & 8 \\ 7 & 1 & 3 \end{vmatrix}$ 10. $\begin{vmatrix} -1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 3 & 2 \end{vmatrix}$

11. $\begin{vmatrix} 6 & 2 & 1 \\ 3 & 0 & 5 \\ 2 & 6 & 4 \end{vmatrix}$ 12. $\begin{vmatrix} 5 & 4 & 0 \\ 1 & 3 & -3 \\ -4 & -2 & 6 \end{vmatrix}$ 13. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 5 & -4 \\ 7 & -1 & 3 \end{vmatrix}$ 14. $\begin{vmatrix} -4 & -1 & 6 \\ 2 & 6 & 4 \\ 3 & 0 & -3 \end{vmatrix}$

317. Multiplying the elements of one column (row) by the minors of the elements of another column (row).—Expanding the determinant at the right we have $a_1A_1 - a_2A_2 + a_3A_3$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Consider now the expression obtained by multiplying a_1 by the minor of b_1 (instead of by the minor of a_1), a_2 by the minor of b_2, and a_3 by the minor of b_3. Then we have

$$a_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix},$$

which by theorem 5, §310 (page 349), is equal to zero. This is a special case of the following general theorem:

If the elements of a column (row) of a determinant are multiplied by the minors of the elements of another column (row) and the sum taken as in expanding a determinant according to minors, the result is equal to zero. That is, $a_1B_1 - a_2B_2 + a_3B_3 = 0$.

EXERCISES

Verify the above theorem by applying it to these determinants.

1. $\begin{vmatrix} 4 & 2 & 3 \\ 6 & 1 & 7 \\ 4 & 1 & 5 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & -3 & 4 \\ 7 & 5 & 2 \\ 3 & 9 & 7 \end{vmatrix}$ 3. $\begin{vmatrix} 5 & 4 & 3 \\ 2 & 7 & 1 \\ 8 & 4 & 6 \end{vmatrix}$ 4. $\begin{vmatrix} 3 & 9 & 11 \\ 2 & 5 & 7 \\ 9 & 1 & 0 \end{vmatrix}$

318. Solving linear equations by means of determinants.—We shall now prove the rule used in §309 for solving three linear equations in three unknowns. Such a set of equations is shown at the right. Use $\triangle$ to represent the determinant obtained by taking the coefficients of x, y, z, in the special relations in which they stand in the equations. Denote the minors of a_1, a_2, a_3 in this determinant by A_1, A_2, A_3 as usual. Multiply equation (1) by A_1, (2) by $-A_2$, and (3) by A_3, and add.

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad (3)$$

$$\triangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $(a_1A_1 - a_2A_2 + a_3A_3)x + (b_1A_1 - b_2A_2 + b_3A_3)y +$
$(c_1A_1 - c_2A_2 + c_3A_3)z = d_1A_1 - d_2A_2 + d_3A_3.$

But by §317, $b_1A_1 - b_2A_2 + b_3A_3 = 0$, $c_1A_1 - c_2A_2 + c_3A_3 = 0$.
Hence $(a_1A_1 - a_2A_2 + a_3A_3)x = d_1A_1 - d_2A_2 + d_3A_3$.
In this equation, the coefficient of x is the Δ of the system. The second member is exactly like the coefficient in the first when a_1, a_2, a_3 are replaced by d_1, d_2, d_3. Hence we have the value of x shown at the right.

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta}$$

EXERCISES

1. In the equations on page 356 multiply (1) by $-B_1$, (2) by B_2, and (3) by $-B_3$ and add, obtaining $(-a_1B_1 + a_2B_2 - a_3B_3)x + (-b_1B_1 + b_2B_2 - b_3B_3)y$, and so on. Show that the coefficients of x and z are zero, and thus obtain the value of y. Remember that $\Delta = -b_1B_1 + b_2B_2 - b_3B_3$.

2. Proceeding as in exercise 1, find the value of z in this same set of equations. (Compare page 346.)

As we shall see, pages 366, 367, the method used here is applicable to a set of n linear equations with n unknowns for all integral values of n.

Solve the following sets of equations:

3. $\begin{cases} 3x + 2y - z = 3 \\ 2x - 3y + 4z = 6 \\ 4x + y - 3z = 5 \end{cases}$

4. $\begin{cases} 4v - 2u + 6w = 12 \\ 9v + 2u - 4w = 16 \\ 3v - u + 8w = 6 \end{cases}$

5. $\begin{cases} 4x + 2y - 3z = 7 \\ 2x - 3y + 2z = 2 \\ -x + 5y - z = 6 \end{cases}$

6. $\begin{cases} 2r - 5s - t = -4 \\ 3r + 2s + 6t = 20 \\ -4r - 5s + 9t = -10 \end{cases}$

7. $\begin{cases} 3a - 9b + 2c - 10 = 0 \\ 5a + 2b + 3c - 12 = 0 \\ 7a + 5b + 4c - 40 = 0 \end{cases}$

8. $\begin{cases} 6x - 3y + 8z - 42 = 0 \\ 2x + 4y + 5z - 50 = 0 \\ 3x - 2y - 4z - 30 = 0 \end{cases}$

9. $\begin{cases} 5u - 9v + 3w - 15 = 0 \\ 7u + 3v + w - 30 = 0 \\ 2u - 4v + 6w - 18 = 0 \end{cases}$

10. $\begin{cases} 5r + 3s - 9t = 12 \\ 3r + 7s - 3t = 6 \\ r - 9s + 12t = 16 \end{cases}$

11. $\begin{cases} 8k - 3l + 8m = 80 \\ 3k + 4l - 4m = 60 \\ 2k - 5l + 9m = 70 \end{cases}$

12. $\begin{cases} 6x - 9y + 12z = 26 \\ -4x + 3y - 5z = 13 \\ -3x - 2y + 15z = 8 \end{cases}$

319. Number of inversions.—The evaluation of determinants of order higher than the third cannot be made by the simple method that we have described. For this reason it is necessary to make a more general study of determinants. The first step in this study is to develop the idea of inversions.

The integers 1, 2, are in their natural order while 2, 1, are in the inverted order. Similarly, 1, 2, 3, 4, 5, are in their natural order, while 1, 4, 3, 5, 2, are partly in the inverted order. The order 1, 4, 3, 5, 2, may be converted into the natural order by repeated interchange of adjacent numerals. If 2 and 5 are interchanged, then 2 and 3, then 2 and 4, and then 4 and 3, we have in succession the orders shown at the right resulting in 1, 2, 3, 4, 5, which is the natural order. This order was obtained from 1, 4, 3, 5, 2 by interchanging adjacent figures four times. This illustrates the following definition.

1 4 3 5 2
1 4 3 2 5
1 4 2 3 5
1 2 4 3 5
1 2 3 4 5

The number of inversions of a given order is the number of times adjacent figures must be interchanged to obtain the natural order.

The number of inversions may be found by taking the sum of the number of figures of lower order that occur after each figure.

Thus, in 1, 4, 3, 5, 2, no figures of lower order occur after 1; two such figures, 3 and 2, occur after 4; one figure of lower order, 2, occurs after 3; and one figure of lower order, 2, occurs after 5.

Hence, $0 + 2 + 1 + 1 = 4$ is the number of inversions in 1, 4, 3, 5, 2.

The following statement is easily verified.

Interchanging any two adjacent figures in an array changes the number of inversions by one.

EXERCISES

Give the number of inversions in each of the following.

1. 1, 3, 5, 2, 4, 6.
2. 2, 4, 6, 1, 3, 5.
3. 6, 5, 1, 2, 4, 3.
4. 4, 2, 6, 5, 3, 1.
5. 3, 4, 1, 6, 5, 2.
6. 5, 2, 6, 4, 3, 1.
7. 2, 3, 6, 1, 5, 4.
8. 1, 6, 2, 4, 5, 3.
9. 3, 4, 1, 2, 5, 6.

10. In 1, 6, 3, 2, 5, 4, verify that by interchanging any two adjacent figures, the number of inversions is changed by one. Also, using the same numbers, change into the natural order by successive interchanging of adjacent figures.

320. General definition of a determinant.—Consider a square array of numbers with n columns and n rows,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ a_{31} & a_{32} & a_{33} \dots a_{3n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} \dots a_{nn} \end{vmatrix}$$

The two subscripts used with each letter indicate the row and the column in which it occurs. Thus, the subscript 11 in a_{11} indicates that a_{11} is in the first row and the first column, and the subscript 32 indicates that a_{32} is in the third row and the second column.

A determinant of the n*th order is a square array of* n^2 *numbers whose numerical value is the algebraic sum of a series of products, each product containing one factor from each column and each row of the square array.*

If in each product the first subscripts of the factors are in natural order, then the products are positive or negative according as the second subscripts have an even or an odd number of inversions.

In this rule zero is regarded as an even number.

Thus, for instance, in the evaluation of

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$a_{11}a_{22}a_{33}$ is positive since the second subscripts are in natural order. That is, the number of inversions is zero.

In the terms $a_{13}a_{21}a_{32}$, and $a_{12}a_{23}a_{31}$, the second subscripts are in the orders 3, 1, 2 and 2, 3, 1, and there are two inversions in each. Hence these terms are both positive.

In $a_{13}a_{22}a_{31}$ there are three inversions in the second subscripts and in $a_{12}a_{21}a_{33}$ and $a_{11}a_{23}a_{32}$ there is one inversion in each, and these terms are all negative.

EXERCISE

Verify, by expanding the determinant at the right, that the definition given above agrees with the definition of a third order determinant given in §308.

$$\begin{vmatrix} 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{vmatrix}$$

321. *Interchanging rows and columns in a determinant.* —In the following determinants the columns and rows are interchanged.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{n3} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{vmatrix}$$

In the expansion of the second determinant each product will appear exactly as in the expansion of the first with the exception that the order of each pair of subscripts will be interchanged. That is, the second determinant will be exactly what the first one would be if the factors of each term were so arranged that the second subscripts would be in their natural order.

Suppose the second subscripts of a term in the first determinant have an even number of inversions. Then rearranging the order of the factors so as to bring the second subscripts into their natural order would cause an even number of inversions in the first subscript. That is, the sign of the term, when considered as a term of the second determinant, would remain positive. If the second subscripts in the first determinant have an odd number of inversions, arranging the factors so as to bring these into natural order would cause an odd number of inversions in the first subscripts and hence the term would remain negative. Hence the two determinants are identical.

We therefore have the following theorem.

If in two determinants the rows of one are the columns of the other and in the same order, the determinants are equal.

322. *Interchanging two columns (or rows).* —Interchanging two adjacent columns in a determinant will interchange the position of two adjacent factors in each term and this will increase or diminish the number of inversions by one. That is, the sign of the term will be changed. Since the sign of each term is changed the sign of the whole determinant is changed.

Hence we have:

Interchanging two adjacent columns (rows) changes the sign of a determinant but leaves its absolute value unchanged.

Suppose now that any two columns (or two rows) are interchanged, and that there are m columns between the two. Then the first column may be placed just before the second by interchanging adjacent columns m times. To place the second column in the place originally occupied by the first, adjacent columns must be now interchanged $m + 1$ times. Hence to interchange any two columns, adjacent columns must be interchanged $2m + 1$ times.

Since $2m + 1$ is an odd number, the required change is affected by interchanging adjacent columns an odd number of times. But each of these changes the sign of the determinant and hence the odd number of changes will change its sign.

Hence we have:

Interchanging any two columns (rows) of a determinant changes its sign but does not change its absolute value.

EXERCISES

1. In $a_{11}\, a_{13}\, a_{14}\, a_{12}\, a_{16}\, a_{15}$ what is the number of inversions of the second subscripts? Interchange a_{11} and a_{16}. What is the number of inversions in this arrangement? Do these results verify the theorem in §321?

2. Carry out in detail the proof of the Theorem in §321. Use a third order determinant written in the terms a_{11}, a_{12}, etc., to guide you in seeing the proof clearly. Also use a fourth order determinant for this purpose.

323. *Multiplying a determinant.*—Since in the expansion of a determinant each product contains one factor from each row and each column, it follows that multiplying each element in a column (row) by a factor k multiplies the whole determinant by k. This holds of course if the multiplier is $1/k$, which represents a division by k. Hence it follows that:

A determinant may be multiplied by a number by multiplying each element in a column (row) by that number.

Thus,

$$2\begin{vmatrix} 4 & 1 & 3 \\ 6 & 2 & 5 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 6 \\ 6 & 2 & 5 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 1 & 3 \\ 12 & 2 & 5 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 3 \\ 6 & 4 & 5 \\ 1 & 6 & 4 \end{vmatrix}.$$

For the same reason we also have:

A determinant may be divided by a number by dividing each element in a column (row) by that number.

324. Identical columns in general determinants. Adding general determinants.—The properties of determinants stated and proved in §§310, 313 hold for determinants in general.

Using §§319, 320 we find that it follows exactly as in these earlier paragraphs that the following theorems hold for any determinant.

1. The value of a determinant remains unchanged when rows are changed into columns and columns into rows.
2. If any two columns (or two rows) are interchanged, the sign of the determinant is changed but not its absolute value.
3. If all the elements in a column (or row) are multiplied or divided by the same number, then the determinant is multiplied or divided by that number.
4. If two columns (or rows) of a determinant are identical, the value of the determinant is zero.

325. Expanding determinants in terms of minors.—The general definition of the minor of an element is exactly that given on page 349 for third order determinants.

Using the notation of §320, we denote the minor of the element a_{11} by A_{11}, the minor of a_{12} by A_{12}, and, in general, the minor of a_{ij} is denoted by A_{ij}.

In the expansion of a determinant the minor of a_{11} is obtained by taking one element from all remaining rows and all remaining columns in all possible combinations. Hence the sum of these products is A_{11}.

That is, the sum of the terms of the expansion containing a_{11} as a factor is $a_{11}A_{11}$.

If the first and second column are interchanged, the element a_{21} is in the first place and by the argument just used its minor consists of A_{21}. Since interchanging the two columns changes the sign of the original determinant, $-A_{21}$ is the minor of a_{21} in that determinant.

In general we can show that the minor of a_{ij} in the original determinant is A_{ij}, the sign being plus or minus according as $i + j$ is even or odd.

From these considerations it is seen that the determinant whose principal diagonal[1] is $a_{11}a_{22} \ldots a_{nn}$ may be written $a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - a_{14}A_{14} + \ldots + (-1)^{n-1}a_{1n}A_{1n}$.

This is said to be an expansion of the determinant using the minors of the elements of the first row.

Similarly, the determinant may be expanded using the minors of the elements of any column or row.

[1]The principal diagonal runs from the upper left corner to the lower right.

326. *Practice in expanding determinants.*—The results of §§323, 324 now enable us to expand any determinant using the method described in §§313, 314.

The process is illustrated in the following examples.

Example 1. Find the value of $\begin{vmatrix} 1 & 3 & 7 & 4 \\ 2 & 4 & 3 & 2 \\ 3 & 7 & 6 & 0 \\ 2 & 4 & 1 & 9 \end{vmatrix}$

$$\begin{vmatrix} 1 & 3 & 7 & 4 \\ 2 & 4 & 3 & 2 \\ 3 & 7 & 6 & 0 \\ 2 & 4 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 & 4 \\ 2 & -2 & 3 & 2 \\ 3 & -2 & 6 & 0 \\ 2 & -2 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 & 4 \\ 2 & -2 & 3 & 2 \\ 1 & 0 & 3 & -2 \\ 2 & -2 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 & 4 \\ 2 & -2 & 3 & 2 \\ 1 & 0 & 3 & -2 \\ 0 & 0 & -2 & 7 \end{vmatrix}$$

The second determinant is obtained from the first by subtracting three times the first column from the second, the third is obtained from the second by subtracting the second row from the third, and the fourth is obtained from the third by subtracting the second row from the fourth. These determinants are equal by §314.

Expanding the last determinant according to the minors of the second column we have

$$-2\begin{vmatrix} 1 & 7 & 4 \\ 1 & 3 & -2 \\ 0 & -2 & 7 \end{vmatrix}.$$

Subtracting the first row from the second gives,

$$-2\begin{vmatrix} 1 & 7 & 4 \\ 0 & -4 & -6 \\ 0 & -2 & 7 \end{vmatrix} = -2\begin{vmatrix} -4 & -6 \\ -2 & 7 \end{vmatrix} = -2[-4\cdot 7 - (-2)(-6)] = 80.$$

The purpose in all these steps is to change the determinant by using §314 so that all elements, except one, in a row or column shall be zero. Expanding in terms of the minors of this row or column will then reduce the order of the determinant by one, when the process may be repeated.

In the determinant at the right the second row may be subtracted from the third and fourth rows at once, thus obtaining the last determinant above without writing the one intervening.

$$\begin{vmatrix} 1 & 0 & 7 & 4 \\ 2 & -2 & 3 & 2 \\ 3 & -2 & 6 & 0 \\ 2 & -2 & 1 & 9 \end{vmatrix}$$

Example 2. Evaluate $\begin{vmatrix} 4 & 6 & 5 & -15 \\ 8 & 9 & 15 & 10 \\ 10 & -12 & 20 & 5 \\ 12 & -6 & 10 & -20 \end{vmatrix}$.

SOLUTION: The factors 2, 3, 5, 5 may be removed from the columns taken in order resulting in the first determinant below.

$$2\cdot3\cdot5^2\begin{vmatrix} 2 & 2 & 1 & -3 \\ 4 & 3 & 3 & 2 \\ 5 & -4 & 4 & 1 \\ 6 & -2 & 2 & -4 \end{vmatrix} = 2^2\cdot3\cdot5^2\begin{vmatrix} 2 & 2 & 1 & -3 \\ 4 & 3 & 3 & 2 \\ 5 & -4 & 4 & 1 \\ 3 & -1 & 1 & -2 \end{vmatrix} = 2^2\cdot3\cdot5^2\begin{vmatrix} 0 & 0 & 1 & 0 \\ -2 & -3 & 3 & 11 \\ -3 & -12 & 4 & 13 \\ 1 & -3 & 1 & 1 \end{vmatrix}$$

The second determinant in this row is obtained from the first by removing the factor 2 from the last row. This determinant may now be reduced by subtracting twice the third column from the first and second, and adding three times the third column to the fourth. This will make all elements in the first row except one equal to zero.

327. *Solving equations stated as a determinant put equal to zero.*

Example. Solve the equation at the right.
SOLUTION: The first step is to expand the determinant. The result is $x^2 - 13x - 7 = 0$, which is an ordinary quadratic.

$$\begin{vmatrix} 1 & x & 3 & 1 \\ 2 & 2 & x & 2 \\ 3 & x & 2 & -1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = 0$$

EXERCISES

Evaluate the following determinants.

1. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 4 & 0 \\ 7 & 1 & 3 & -3 \\ 2 & -4 & -2 & 6 \end{vmatrix}$ 2. $\begin{vmatrix} 4 & 1 & 0 & 6 \\ 2 & 6 & 4 & 10 \\ 3 & 0 & 4 & -3 \\ 6 & 1 & 4 & 2 \end{vmatrix}$ 3. $\begin{vmatrix} 4 & 5 & 2 & 9 \\ 3 & -1 & 4 & 6 \\ 4 & 5 & -6 & 2 \\ 8 & 1 & 0 & 3 \end{vmatrix}$ 4. $\begin{vmatrix} a & a^2 & a^3 & a^4 \\ 2a & a & 3a & -a \\ 6 & -4 & 4 & 6 \\ 2 & -3a & 2 & 6 \end{vmatrix}$

5. $\begin{vmatrix} 2x & -x & x^2 & -x^2 \\ x & -3x & 2x & 6 \\ 4 & x & 6 & -x \\ x & -3 & x & -2 \end{vmatrix}$ 6. $\begin{vmatrix} 6 & -b & 2 & 6 \\ 3 & b^2 & -4 & 0 \\ -2 & -b^2 & 11 & -4 \\ 7 & b & 5 & 3 \end{vmatrix}$ 7. $\begin{vmatrix} 3 & 4 & 2 & 6 \\ -4 & 5 & 7 & -4 \\ -2 & -7 & 2 & 8 \\ -6 & 4 & -8 & 0 \end{vmatrix}$ 8. $\begin{vmatrix} 9 & -4 & 7 & 2 \\ a & b & c & d \\ 8 & -2 & 0 & 6 \\ 5 & 3 & 2 & 0 \end{vmatrix}$

9. $\begin{vmatrix} 7 & b & c & 2 \\ b & 2 & 1 & c \\ 2 & 3 & 0 & 2 \\ 3 & c & 9 & b \end{vmatrix}$ 10. $\begin{vmatrix} a & 2a & 3a & 4a \\ 4 & 2 & 5 & 3 \\ b & b^2 & b^3 & b^4 \\ 3a & 4a^2 & 5a^2 & 6a^3 \end{vmatrix}$ 11. $\begin{vmatrix} 2m & 3n & 6m & 8n \\ 5m & 4n & 5m & 2n \\ -3m & -2n & -3m & 5n \\ m & n & m & n \end{vmatrix}$

Solve the following equations for x.

12. $\begin{vmatrix} 4 & x & 2 & 4 \\ 2 & 7 & -x & 2 \\ 5 & -x & 7 & 4 \\ 3 & 8 & x & 3 \end{vmatrix} = 0$ 13. $\begin{vmatrix} 2 & 3 & 7 & 8 \\ x & 4 & -2 & 0 \\ 7 & 5 & x & 9 \\ 8 & 2 & 0 & 4 \end{vmatrix} = 0$ 14. $\begin{vmatrix} x & 2x & 3 & -x \\ 2x & 4 & 5 & 0 \\ 3 & 9 & 8 & x \\ -3x & 2 & 9 & -2x \end{vmatrix} = 0$

15. $\begin{vmatrix} 2x & 5 & 6 & x \\ 3 & 2 & 2 & 6 \\ x & 3 & 6 & 3x \\ 5 & 2 & 4 & 1 \end{vmatrix} = 0$ 16. $\begin{vmatrix} 3 & x & 5 & 2 \\ 2 & 3 & 4 & 6 \\ 7 & 0 & 2x & 4 \\ 1 & 0 & 6 & 3 \end{vmatrix} = 0$ 17. $\begin{vmatrix} 2 & 3x & 2x & 4 \\ x & 4 & 0 & 5 \\ 8 & 2 & 3 & 7 \\ 3 & 8 & 0 & 1 \end{vmatrix} = 0$

328. ***Multiplying the elements of one column (row) by the minors of corresponding elements of another column (row).***—On comparing the expressions

$$a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - a_{41}A_{41} + \cdots$$
$$a_{11}B_{11} - a_{21}B_{21} + a_{31}B_{31} - a_{41}B_{41} + \cdots$$

we note that if b's instead of a's are used in the second expression, we have the expansion of the general determinant in terms of minors of the elements in its second column. Hence this second expression represents the expansion of the general determinant in which both the first and second columns are identical, in which, namely, both columns consist of a's. The value of this expression is therefore zero and we have the theorem:

If the elements of a column (row) of a determinant are multiplied by the minors of the elements of another column (row) and the sum taken as in expanding a determinant according to minors, the result is equal to zero.

This is the theorem proved in §317 for determinants of the third order.

EXERCISE

Verify the above theorem as follows.

Using the first determinant at the right, write the expansion in terms of the elements in the first column and their minors. Then replace A_1, A_2, A_3, A_4, by B_1, B_2, B_3, B_4, giving $a_1B_1 - a_2B_2 + a_3B_3 - a_4B_4$. Then show that the result is the second determinant at the right.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \qquad \begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_2 & a_2 & c_2 & d_2 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

329. Solving equations by determinants.—We shall now repeat the argument in §318 using four equations:

$$a_1x + b_1y + c_1z + d_1v = f_1 \quad (1)$$
$$a_2x + b_2y + c_2z + d_2v = f_2 \quad (2)$$
$$a_3x + b_3y + c_3z + d_3v = f_3 \quad (3)$$
$$a_4x + b_4y + c_4z + d_4v = f_4 \quad (4)$$

Denote the determinant at the right by $\triangle$. Multiply equations (1), (2), (3), (4), respectively, by A_1, $-A_2$, A_3, $-A_4$, of this determinant and add. Then we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$(a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4)x + (b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4)y$$
$$+ (c_1A_1 - c_2A_2 + c_3A_3 - c_4A_4)z + (d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4)v$$
$$= f_1A_1 - f_2A_2 + f_3A_3 - f_4A_4.$$

By §328 the coefficients of y, z, v, are equal to zero, while the coefficient of x is $\triangle$, and the right member is $\triangle$ with the a's replaced by f's.

Hence,

$$x = \frac{\begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ f_3 & b_3 & c_3 & d_3 \\ f_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\triangle}.$$

To solve for y multiply the equations by $-B_1$, $+B_2$, $-B_3$, $+B_4$ respectively; to solve for z, multiply the equations by C_1, $-C_2$, C_3, $-C_4$; and to solve for v, multiply by $-D_1$, D_2, $-D_3$, D_4.

Proceeding in this manner we obtain:

$$y = \frac{\begin{vmatrix} a_1 & f_1 & c_1 & d_1 \\ a_2 & f_2 & c_2 & d_2 \\ a_3 & f_3 & c_3 & d_3 \\ a_4 & f_4 & c_4 & d_4 \end{vmatrix}}{\triangle}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & f_1 & d_1 \\ a_2 & b_2 & f_2 & d_2 \\ a_3 & b_3 & f_3 & d_3 \\ a_4 & b_4 & f_4 & d_4 \end{vmatrix}}{\triangle}, \quad v = \frac{\begin{vmatrix} a_1 & b_1 & c_1 & f_1 \\ a_2 & b_2 & c_2 & f_2 \\ a_3 & b_3 & c_3 & f_3 \\ a_4 & b_4 & c_4 & f_4 \end{vmatrix}}{\triangle}.$$

Note again how $\triangle$, the determinant of the system of equations, is obtained, and how the numerators in the values of x, y, z, v, are obtained from $\triangle$ by replacing a's, b's, c's, d's successively by the f's.

This process is used without modification in solving n linear equations. For convenience denote the unknowns by $x_1, x_2, \ldots, x_n$.

Then the equations are:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2 \quad (2)$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = c_n \quad (n)$$

Multiplying these equations in order by $A_{11}, -A_{21}, A_{31}, \dots, +(-1)^{n-1}A_{n1}$, adding, and dividing by the coefficient of x_1, we have

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & \dots & a_{1n} \\ c_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ c_n & a_{n2} & \dots & a_{nn} \end{vmatrix}}{\Delta},$$

and so on for $x_2, x_3, \dots, x_n$.

EXERCISES

Solve the following sets of equations.

1. $\begin{cases} x + y + z + v = 6 \\ 3x - y + 2z - 3v = 6 \\ -x + 6y - 3z + 2v = 2 \\ 2x - 3y + 4z - 5v = 8 \end{cases}$

2. $\begin{cases} 2x - 2y + 3z + 4v = 6 \\ 4x + 3y - 2z + 8v = 3 \\ 6x - 5y + 2z - 12v = 3 \\ 2x + y + 4z - 16v = 6 \end{cases}$

3. $\begin{cases} 2x - y + z - v = 3 \\ 7x + 2y - 3z + 2v = 18 \\ -x - 3y + 8z - 2v = 14 \\ 3x + y - 4z + 7v = 18 \end{cases}$

4. $\begin{cases} 4x - 3y + 6z = 8 \\ 2x + 2z - 4v = 12 \\ 7x - y + 2v = 5 \\ -3y + 7z - 3v = 10 \end{cases}$

5. $\begin{cases} x - y + z - u + v = 12 \\ 2x + y - 2z + 3u - v = 16 \\ -3x + 2y + z - 4u + 3v = 8 \\ -x + 6y - z + 8u - 2v = 10 \\ 2x + 4y - 2z - 6u + 2v = 24 \end{cases}$

6. $\begin{cases} 2y - 3z + 2u - v = 27 \\ 4x + z + 4u - 2v = 18 \\ -3x + 2y - 3u + v = 12 \\ -x + 3y - 2z + 4v = 9 \\ 3x - 5y + 7z - 5u = 24 \end{cases}$

7. $\begin{cases} 2x - y + 3z - u + 4v = 15 \\ x + 4y - 3z + 2u - v = 18 \\ 7x - 5y + 2z - u + 3v = 21 \\ -3x + 2y - z + 3u - 2v = 6 \\ 2x - 3y - 4z + 8u + 6v = 7 \end{cases}$

8. $\begin{cases} 5x - 2y - z + 2v = 30 \\ x + 5y + u - 3v = 25 \\ 6x + 2z + 3u - 3v = 20 \\ 2x + 2y - 3z + 5u = 15 \\ 3y - 5z + 6u + 2v = 10 \end{cases}$

330. Consistent independent, dependent, and inconsistent equations.—If the determinant $\triangle$ of a system of equations is different from zero the equations are satisfied by one and only one set of values of the unknowns. The equations of such a set are independent and consistent (see §113).

If $\triangle = 0$, either one of two different cases may arise. If the numerators are also equal to zero, the values of the unknowns obtained from the determinant solution are in the form $\frac{0}{0}$. In this case the equations are satisfied by an indefinite number of sets of values, and the equations are dependent (and at the same time consistent).

In case $\triangle = 0$ while one or more of the numerators are different from zero, there are no values which satisfy the equations, and the equations are inconsistent (and also independent).

EXERCISES

Determine whether the equations in each of the following sets are independent and consistent, dependent, or inconsistent. Solve those that are independent and consistent.

1. $\begin{cases} 6x + y + 2z = 7 \\ 3x - y + 4z = 14 \\ 5x + 2y - 3z = -7 \end{cases}$

2. $\begin{cases} x + 3y + 4z = 2 \\ 4x + 12y + 16z = 8 \\ 3x + 9y + 12z = 6 \end{cases}$

3. $\begin{cases} 5x - y - z = 18 \\ x + 4y + 2z = -7 \\ 3x + 2y - 4z = 9 \end{cases}$

4. $\begin{cases} 2x + 3y + z = 7 \\ 4x + 6y + 2z = 1 \\ 8x + 12y + 4z = 6 \end{cases}$

5. $\begin{cases} x + 2y + 3z = 1 \\ 3x + 6y + 9z = 3 \\ 5x + 10y + 15z = 5 \end{cases}$

6. $\begin{cases} x + 4y - z = 4 \\ 4x + 16y - 4z = 2 \\ 7x + 28y - 7z = 6 \end{cases}$

7. $\begin{cases} 3x + 4y + 2z = 5 \\ x + 2y - 3z = -17 \\ 2x - y + 4z = 4 \end{cases}$

8. $\begin{cases} 6x + 2y + 4z = 1 \\ 3x + y + 2z = 8 \\ 12x + 4y + 8z = 6 \end{cases}$

9. $\begin{cases} 2x + 3y - z - 2w = -1 \\ x + 2y + 3z - w = -4 \\ 4x + 8y - 5z + w = 16 \\ x + y + z + w = 4 \end{cases}$

10. $\begin{cases} x + 3y - 2z - w = 7 \\ 2x + 6y - 4z - 2w = 4 \\ 5x + 15y - 10z - 5w = 3 \\ 3x + 9y - 6z - 3w = 8 \end{cases}$

331. A set of n + 1 *equations containing* n *unknowns.*—Let us consider the four equations,

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad (3)$$
$$a_4x + b_4y + c_4z = d_4. \quad (4)$$

Solving equations (1), (2), (3) simultaneously, we have

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta},\ y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta},\ z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta},\text{ where } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Substituting these values in (4), interchanging columns as shown, clearing of fractions, and transposing, we have

$$a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} - b_4 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Hence by §317,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0,$$

which is the condition that the four given equations shall be consistent.

A similar condition is sufficient for the consistency of $n + 1$ equations in n unknowns.

EXERCISES

1. Prove that sets 1 and 2 below are consistent.
2. Find the values of a that will make sets 3 and 4 consistent.

1. $\begin{cases} 7x - 3y = 5 \\ 5x + 2y = 16 \\ 2x + 5y = 19 \end{cases}$

3. $\begin{cases} 2x - 2y = 5 \\ 7x + ay = 18 \\ 3x - 2y = 10 \end{cases}$

2. $\begin{cases} 4x - 2y - 3z = 2 \\ 3x + 5y + z = -5 \\ x - 3y + 5z = 31 \\ -2x + 6y + 2z = -14 \end{cases}$

4. $\begin{cases} ax - 3y + 2z = 10 \\ 2x + y - 4z = 6 \\ -3x + 6y + 3z = 14 \\ 5x - 6y + 4z = 20 \end{cases}$

332. Homogeneous equations.—An equation of the type

$$a_1x + b_1y + c_1z = 0$$

is said to be a linear homogeneous equation.

It is evident that the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

is satisfied by $x = 0, y = 0, z = 0$; but it is not at all certain that it is satisfied by any other values of the unknowns.

Suppose that this system is satisfied by a set of values, at least one of which, as z for instance, is different from zero. Then the equations may be divided by z, reducing them to three equations in the two unknowns x/z and y/z.

By §331, the condition that these equations shall be consistent is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Clearly this argument applies to a set of n such equations, and we have the theorem:

If the determinant formed by the coefficients of a set of linear homogeneous equations is equal to zero, the equations are satisfied by a set of values not all of which are zero, and otherwise not.

EXERCISES

1. Decide whether or not systems 1, 2, and 3 below are satisfied by any set of values not all of which are zero.

2. In systems 4, 5, and 6 find the values of k which will make these systems satisfied by a set of values not all of which are zero.

1. $\begin{cases} 4x - 2y + 3z = 0 \\ x + 5y - 2z = 0 \\ -3x + 3y + 5z = 0 \end{cases}$

4. $\begin{cases} 8x - 5y + 3z = 0 \\ 2x + 4y + kz = 0 \\ -4x + 2y - 5z = 0 \end{cases}$

2. $\begin{cases} 2x + 3y - 5z = 0 \\ x - 9y + 8z = 0 \\ 4x + 2y - 3z = 0 \end{cases}$

5. $\begin{cases} 3x - 3y + kz = 0 \\ -2x + 2y - 5z = 0 \\ 8x + 5y + 6z = 0 \end{cases}$

3. $\begin{cases} 4x - y + 2z = 0 \\ x + 2y - z = 0 \\ 3x - 3y + 2z = 0 \end{cases}$

6. $\begin{cases} kx + 2y - z = 0 \\ x - 3y + kz = 0 \\ 3x + 2y + z = 0 \end{cases}$

CHAPTER 26:

CUMULATIVE REVIEWS

CUMULATIVE REVIEW I

In case no directions are given, perform all indicated operations and solve all equations.

(After p. 68)

1. From the sum of $x^2 - 3$ and $4x^3 - 3x^2 + 7x + 5$ subtract $2x^3 + 7x^2 - 9$.
2. Multiply $x^4 + x^3 + x^2 + x + 1$ by $x - 1$.
3. Factor $a^2b^4 - c^4b^2$.
4. Factor $4m^2 + 20mn + 25n^2$.
5. Simplify $\dfrac{x - 1}{2x - \dfrac{3x}{x + 2}}$.
6. $\dfrac{x^2 - x - 6}{x^2 + 2x - 3} \cdot \dfrac{x^2 - 4x + 3}{x^2 + 3x + 2} \div \dfrac{x^2 - x - 6}{x^2 + 2x - 3}$.

CUMULATIVE REVIEW II

(After p. 79)

1. From $9x^2 - 3x + 7$ subtract $3x^2 - 2x^3 - 4$ and then add $x^4 - 9x^2 + 2$.
2. Factor $4a^2 - 12ab + 9b^2$.
3. Solve $(x - 1)^2 + (x + 1)^2 = (2x - 1)(x + 2) + 7$.
4. Solve $(x - 3)(x + 5) + 7x - x^2 = 3$.
5. Solve $\dfrac{2}{x + 1} + \dfrac{3}{x - 2} = \dfrac{5}{x + 2}$.
6. Solve $\dfrac{2x^2 - 3x + 7}{x^2 - 4} = 2$.

CUMULATIVE REVIEW III

(After p. 83)

1. Subtract $a^3 - 3ab + 2b^3$ from $8ab - a^3 + b^2$ and then add $5a^2b + 2ab - 3b^2$.
2. Factor $x^2 - 3x - 28$.
3. Solve $(2x - 3)(x + 2) + 12x = 2x^2 - 19$.
4. Solve $\dfrac{1}{x + 1} + \dfrac{1}{x - 1} = \dfrac{4}{x^2 - 1}$.
5. Solve $\dfrac{2}{x^2 + 2x - 24} + \dfrac{1}{x^2 + 11x + 30} = \dfrac{3}{x^2 + 3x - 18}$.
6. Solve $d_1w_1 + d_2w_2 = d_3w_3$ for each letter in terms of the others.

CUMULATIVE REVIEW IV

(After p. 96)

1. Multiply $x^5 - x^4 + x^3 - x^2 - x + 1$ by $x + 1$.
2. Solve $(1 - 3x)(1 - x) - 4x = 3x^2 - 31$.
3. Solve $\frac{4}{x+5} + \frac{10}{x^2 - 25} - \frac{1}{x-5} = 0$.
4. Find the approximate solution of $3x - 2y = 6$ and $x + 3y = 4$ by constructing graphs.
5. Find the approximate solution of $x - 3y = 2$ and $3x + y = 4$ by constructing graphs.

CUMULATIVE REVIEW V

(After p. 113)

1. Factor $a^2 - 4a - 32$.
2. Solve $\frac{x-3}{x^2 - 6x + 5} - \frac{2(3x-1)}{3x^2 - 13x - 10} + \frac{3x+1}{3x^2 - x - 2} = 0$.

3. Find the approximate solution of $2y - 3x - 4 = 0$ and $x - 2y - 8 = 0$ by constructing graphs.
4. Solve $2x - 7y + 2 = 0$ and $2y + 4x + 3 = 0$.
5. Solve $3x - 2y - 9 = 0$, $x - 3y + 4 = 0$.
6. Evaluate $\frac{a + 2b}{a - 3b}$ for $a = \frac{2}{3}$, $b = \frac{3}{4}$.

CUMULATIVE REVIEW VI

(After p. 120)

1. Divide $2x^3 - 9x^2 + 16x - 15$ by $2x - 5$.
2. Solve $A = p + prt$ for each letter in terms of the other.
3. Solve $3y + 2x - 8 = 0$, $4x - 3y - 10 = 0$.
4. Solve $2a - 3b + 18 = 0$, $2b + 7a - 10 = 0$.
5. Solve $(6x - 5)(x - 1) - (2x - 1)(3x + 1) = -30$.
6. Find the approximate solution of $3y + x - 8 = 0$ and $2y - 5x + 3 = 0$ by constructing graphs.

CUMULATIVE REVIEW VII

(After p. 120)

1. Factor $x^4 + 5x^2 + 6$.
2. Solve $(x + 1)^2 + (x - 1)^2 = (2x - 1)(x + 2) + 10$.
3. Solve $6m - n + 8 = 0$, $2m + 5n = 20$.
4. Solve $x + y + z = 8$, $x - y + z = 6$, $x - y - z = 2$.
5. Simplify $\sqrt{12}$, $\sqrt{50}$, $\sqrt{75}$, $\sqrt{125}$.
6. Simplify $\sqrt{(a^2 - 9)(a + 3)}$, $\sqrt{a^3b^2c(a - b)^3}$.

CUMULATIVE REVIEW VIII

(After p. 126)

1. Factor $a^4 - 7a^2 + 6$.
2. Solve $\dfrac{4x+1}{2x^2+3x+1} - \dfrac{2x+3}{2x^2-13x-7} - \dfrac{x+4}{x^2-6x-7} = 0$.
3. Solve $2x - 3y + z - 10 = 0$, $x + 2y - 2z - 6 = 0$, $-2x - y + 8z - 2 = 0$.
4. Solve $3x - 2y + z = 10$, $2x + 2y - 3z = 8$, $x - y + 4z = 2$.
5. Simplify $\sqrt{(a+b)^3(a-b)^3}$, $\sqrt{32(a^2-b^2)(a+b)}$.
6. Simplify $\sqrt{pq^3(p-q)^3}$, $\sqrt{m^3n^2(m+n)^5}$.

CUMULATIVE REVIEW IX

(After p. 126)

1. Divide $6a^3 - 13a^2b + 9ab^2 - 2b^3$ by $3a - 2b$.
2. Solve $(5x-1)(x+3) + (3x-2)(x+3) = (4x-1)(2x+3) + 27$.
3. Solve $\dfrac{2}{x^2-2x-3} - \dfrac{1}{x^2-x-6} = \dfrac{1}{x^2+x-12}$.
4. Solve $5x + y - z = 6$, $2x + 3y - 3z = 8$, $x - y + 4z = 2$.
5. Solve $2a - 3b + 7c - 20 = 0$, $a - b + 5c = 14$, $a - 2b = 7 + c$.
6. $\left(\dfrac{a}{a+b} - \dfrac{b}{a-b}\right) \div \left(\dfrac{a}{a-b} + \dfrac{b}{a+b}\right)$.

CUMULATIVE REVIEW X

(After p. 129)

1. Factor $a^3 - 8$, $a^3 + 8$.
2. Solve $A = \dfrac{1}{2}(a+b)$ for a and for b.
3. Solve $3m - 2n - p = 3$, $m + 2n + 2p = 15$, $-m + 3n - p = 4$.
4. Simplify $\sqrt{(a-1)(a^2-3a+2)}$, $\sqrt{(x^2-5x+6)(x^2-4x+3)}$.
5. Simplify $\sqrt{\dfrac{1}{2}}$, $\dfrac{1}{\sqrt{2}}$, $\sqrt{\dfrac{2}{3}}$, $\sqrt{\dfrac{3}{4}}$.
6. Multiply $x - 2\sqrt{x} + 1$ by $x + 2\sqrt{x} + 1$.

CUMULATIVE REVIEW XI

(After p. 129)

1. Factor $27 - x^3$, $125 - a^3$.
2. $4(2x^2 + 1) - 8(x+1)^2 = 3x - 4$.
3. $6x - 3y + 2z = 12$, $2x + 4y + 5z = 26$, $x + 3y + 5z = 18$.
4. Simplify $\sqrt{\dfrac{a}{b}}$, $\sqrt{\dfrac{a}{b^3}}$, $\sqrt{\dfrac{a^3}{b^3}}$.

5. Multiply $x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}$ by $x^{1/3} - y^{1/3}$.
6. Change so as to make all exponents positive
$$\frac{x^2y^{-3}p^{-2}}{a^2b^{-1}}, \qquad \frac{a^{-1}b^2c^{-2}}{a^{-3}b^{-1}c^3}.$$

CUMULATIVE REVIEW XII

(After p. 129)

1. Factor $a^4 - 1$, $a^6 - 1$.
2. Solve $M = \dfrac{a+b+c}{3}$, for a, b, and for c.
3. Solve $2p - 3q + 7r - 12 = 0$, $4p + q + 3r = 8$, $p + 7q - 5r = 6$.
4. Simplify $\sqrt{\dfrac{a}{a+b}}$, $\dfrac{x}{\sqrt{x+y}}$, $\sqrt{\dfrac{3}{(x-y)^3}}$.
5. Change so as to make all exponents positive
$$\frac{(a-b)^{-1}}{a+b}, \qquad \frac{x^{-1}yz^{-2}}{3a^{-3}b^{-2}}.$$

6. What is a rationalizing factor of $\sqrt{a} + \sqrt{b}$? of $\sqrt{a} - \sqrt{b}$?

CUMULATIVE REVIEW XIII

(After p. 138)

1. Divide $x^4 + x^3 - 5x^2 - 7x - 2$ by $x^2 - 2x - 1$.
2. Add $\dfrac{x+1}{x^2 - 3x + 2}$ and $\dfrac{2x-3}{x^2-4}$.
3. Simplify $\sqrt{\dfrac{2}{(x^2-y^2)(x+y)}}$, $\sqrt{\dfrac{3a^2b}{5(a-b)^3}}$.
4. Multiply $x^{1/4} - y^{1/4}$ by $x^{1/4} + y^{1/4}$ and the product by $x^{1/2} + y^{1/2}$.
5. Change so as to make all exponents positive
$$\frac{c^0p^{-3}(r-s)^{-1}}{(a+b)^{-1}}, \qquad \frac{p^{-2}qr^{-1}}{s^{-2}t^{-1}}.$$
6. Rationalize the denominator in $\dfrac{a}{\sqrt{a}+\sqrt{b}}$, $\dfrac{a}{\sqrt{a}-\sqrt{b}}$.

CUMULATIVE REVIEW XIV

(After p. 138)

1. Add $\dfrac{1}{x^2+x-6}$, $\dfrac{x+1}{x^2+2x-3}$ and $\dfrac{x+2}{x^2-3x+2}$.
2. Solve $\dfrac{x-3}{x+6} + \dfrac{x+4}{x-6} - \dfrac{2x^2}{x^2-36} = 0$.
3. Use negative exponents instead of the fractional form
$$\frac{2abc^3}{m^2n^3}, \qquad \frac{5rst^2}{c^{-1}d^3p^2}.$$

4. Rationalize the denominator in $\frac{3}{\sqrt{3}-\sqrt{2}}, \frac{4}{\sqrt{3}+\sqrt{2}}$.

5. Rationalize the denominator in $\frac{4}{\sqrt{6}-\sqrt{2}}, \frac{4}{\sqrt{6}+\sqrt{2}}$.

6. Rationalize the denominator in

$$\frac{a}{\sqrt{a+b}-\sqrt{a-b}} \text{ and } \frac{a}{\sqrt{a+b}+\sqrt{a-b}}.$$

CUMULATIVE REVIEW XV

(After p. 148)

1. Add $\frac{1}{a^2-b^2}, \frac{1}{(a+b)^2}$ and $\frac{1}{(a-b)^2}$.

2. Solve $2u-3v-w+2=0$, $3u+2v+w=8$, $u+5v-2w=4$.

3. Simplify $\sqrt{\frac{3x^2+2x^3+x^4}{(x-2)^2(x^2-4)}}, \sqrt{\frac{18a^3b^5}{(a-b)^3(a^2-b^2)}}$.

4. Multiply $a^{3/4}+a^{1/2}b^{1/4}+a^{1/4}b^{1/2}+b^{3/4}$ by $a^{1/4}-b^{1/4}$.

5. Use negative exponents instead of writing the expressions in fractional form

$$\frac{a^0b^3c}{b^0p^8}, \qquad \frac{5xy}{(a-b)^3}.$$

6. Rationalize the numerator in $\frac{\sqrt{7}-\sqrt{3}}{4}, \frac{\sqrt{a}-\sqrt{b}}{3}$.

CUMULATIVE REVIEW XVI

(After p. 148)

1. $(7x+2)(x-3)-7x^2+5x=-34$.

2. $8x-9y+3z=32$, $2x+5y-2z=24$, $3x+2y-8z=-12$.

3. Use negative exponents instead of writing the expressions in fractional form

$$\frac{4k^3l^4m^2}{5^0a^3b^2c}, \qquad \frac{abc}{(a-b)^2(a+b)}.$$

4. Rationalize the numerators in

$$\frac{\sqrt{x+1}-\sqrt{x-1}}{x+1} \text{ and } \frac{\sqrt{x+1}+\sqrt{x-1}}{x-1}.$$

5. Add $\frac{7}{2x^2+3x-2}, \frac{5}{2x^2+5x+2}$ and $\frac{3}{4x^2-1}$.

6. Solve $2ax^2-bx+c=0$, $2ax^2+bx-c=0$.

CUMULATIVE REVIEW XVII

(After p. 148)

1. Solve for x. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}+\frac{d}{x}=e.$
2. Reduce to the form $P+Qi$ the product $(3+2i)(5-4i)$.
3. Reduce to the form $P+Qi$ the product $(a+bi)(a-bi)(a+bi)^2$.
4. Add $\frac{3a-1}{a^2-10a+21}$, $\frac{a+1}{a^2-a-6}$ and $\frac{a-1}{a^2-5a-14}$.
5. Solve $x^2-4x-7=0$, $x^2+4x=7$.
6. Solve $5x^2-3x+2=0$, $5x^2+3x-2=0$.

CUMULATIVE REVIEW XVIII

(After p. 150)

1. Multiply x^2-5x+6 by x^2-4 and divide the product by x^2-x-6.
2. Factor $a^2+4a+4-9x^2$.
3. Solve $(a+b)x^2-(a-b)x+ab=0$.
4. Solve $(a-b)x^2+(a+b)x+2ab=0$.
5. Reduce to the form $P+Qi$ the product $(3a+3bi)(a-7bi)$.
6. Rationalize the numerators $\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{x+4}$, $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{x-4}$.

CUMULATIVE REVIEW XIX

(After p. 157)

1. Solve $\frac{3x-7}{x-4}+\frac{5x-4}{x+3}-\frac{8x^2-3}{x^2-x-12}=0$.
2. Reduce to the form $P+Qi$ the product $(x+yi+3)(x+y+4i)$.
3. Without solving determine the character of the roots of $3x^2-9x+10=0$, $x^2+7x+8=0$.
4. Solve $x^2+x+1=0$ and find the square of each root.
5. From graphs find approximate solutions of $y=x^2$, $2x-3y=6$.
6. For what values of a is $x-y=a$ tangent to $x^2+y^2=16$?

CUMULATIVE REVIEW XX

(After p. 171)

1. Evaluate $\frac{x^2-y^2}{x^2+y^2}$ for $x=\frac{1}{2}$, $y=\frac{1}{3}$.
2. Rationalize the numerator in $\frac{\sqrt{x^2-3x+1}-\sqrt{x^2+3x-1}}{\sqrt{x+1}-\sqrt{x-1}}$.
3. Without solving determine the character of the roots of $2x^2-9x-3=0$, $2x^2+9x-3=0$.

4. Find the product $\left(\frac{1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)$.

5. Find by constructing graphs the approximate solutions of $y = x^2 - x + 1$, $x + 3y = 8$.

6. Solve $x^2 + y^2 = 25$, $25x^2 - 9y^2 = 225$.

CUMULATIVE REVIEW XXI

(After p. 171)

1. Rationalize the numerator in $\frac{\sqrt{(a+x)^2+b^2} - \sqrt{a^2+b^2}}{a+b}$.

2. Reduce to the form $P + Qi$ the product $(x^2 - 3xi + 2)(xi - 7)$.

3. Without solving determine the character of the roots of $7x^2 + 5x - 15 = 0$, $7x^2 - 5x + 15 = 0$.

4. Reduce $\frac{-1+\sqrt{3}i}{-1-\sqrt{3}i}$ to the form $P + Qi$.

5. Find a so that $x - y = a$ is tangent to $y = x^2 - 3x + 1$.

6. Solve $2x - 5y - 9 = 0$, $x + 3y + 10 = 0$ by constructing graphs.

CUMULATIVE REVIEW XXII

(After p. 171)

1. Solve $(x+1)^2 + (x+2)^2 + (x+3)^2 = 3(x-1)(x+4) + 41$.

2. Reduce to the form $P + Qi$ $\frac{a+bi}{a-bi}$, $\frac{a-bi}{a+bi}$.

3. Find by constructing graphs the approximate solutions of $y = x^2 + x - 4$, $x - y = 6$.

4. Evaluate $\frac{a^2-3a+1}{a^2+2a-2}$ for $a = \frac{3}{4}$.

5. Find a so that $x + y = a$ is tangent to $y = x^2 + 2x - 4$.

6. Simplify $\left(\frac{4^{-2}x^{-3}}{y^{-2}}\right)^2$, $\left(\frac{x^2y^2}{x^{2/3}y^{1/2}}\right)^{-1/2}$.

CUMULATIVE REVIEW XXIII

(After p. 171)

1. Reduce to the form $P + Qi$ $\frac{3-7i}{5+2i}$, $\frac{3+7i}{5-2i}$.

2. Without solving determine the character of the roots of $2x^2 - 9x + 60 = 0$, $2x^2 + 9x - 60 = 0$.

3. Reduce $-\frac{-1-\sqrt{3}i}{1+\sqrt{3}i}$ to the form $P + Qi$.

4. Solve $x^2 + y^2 = 25$, $\frac{x^2}{36} + \frac{y^2}{16} = 1$.

5. Solve $x^2 + y^2 = 25, \dfrac{x^2}{36} - \dfrac{y^2}{16} = 1.$

6. Evaluate $\dfrac{x^4 - x^2y + y^4}{x^4 + x^2y^2 + y^4}$ for $x = \frac{1}{2}, y = \frac{1}{4}$.

CUMULATIVE REVIEW XXIV

(After p. 184)

1. Subtract $x^2 - 3x + 7$ from $5x - 2$ and multiply the remainder by $x^2 - 7x + 1$.

2. Reduce to the form $P + Qi$ $\dfrac{a + b + 3i}{a - b - 3i}$.

3. Without solving determine the character of the roots of $7 - 8x + 3x^2 = 0, -7 + 8x + 3x^2 = 0$.

4. Find the sum and the product of the roots in $2x^2 - 8x + 3 = 0$, $2x^2 + 8x - 3 = 0$.

5. Evaluate $\dfrac{x^4 - y^4}{x^4 + y^4}$ for $x = \frac{1}{2}, y = \frac{1}{3}$.

6. Find a so that $3y = a$ is tangent to $y = x^2 - 4x + 7$.

CUMULATIVE REVIEW XXV

(After p. 184)

1. Evaluate $\dfrac{2m^2 - mn + n^2}{m^2 + mn - n^2}$ for $m = \frac{2}{3}, n = \frac{1}{2}$.

2. Find a so that $x + y = a$ is tangent to $x^2 + y^2 = 16$.

3. Solve $x^2 + y^2 = 25, y = x^2$.

4. Solve $x^2 + y^2 = 25, y = x^2 - 5$.

5. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{7}{8}$, find the value of $\dfrac{a + c + e}{b + d + f}$.

6. Simplify $\dfrac{\dfrac{1}{x^2 - y^2} + \dfrac{1}{x - y} - \dfrac{1}{x + y}}{\dfrac{1}{x - y} + \dfrac{1}{x + y}}$.

CUMULATIVE REVIEW XXVI

(After p. 196)

1. Reduce $\dfrac{5 - 2a + 3i}{6 + 3a - 5i}$ to the form $P + Qi$.

2. Find the sum and the product of the roots in $x^2 - 9x - 10 = 0$, $x^2 + 9x + 10 = 0$.

3. Find by constructing graphs the approximate solutions of $y = x^2 - 4x - 2$, $2x - y = 4$.

4. Find a so that $x + 2y = a$ is tangent to $x^2 + y^2 = 9$.
5. Solve $x^2 + y^2 = 25$, $y = x^2 + 5$.
6. Simplify $\dfrac{1 - \dfrac{b}{a - \dfrac{a^2 - b^2}{a + b}}}{1 + \dfrac{a}{1 + \dfrac{b^2 - a^2}{a + b}}}$.

CUMULATIVE REVIEW XXVII

(After p. 196)

1. Reduce $\dfrac{x + y + zi}{x - y - zi}$ to the form $P + Qi$.
2. Factor $2x^2 - 5x + 2$, $2x^2 + 3x - 2$.
3. Simplify $\dfrac{x - 1}{2x - \dfrac{3x}{x + 2}}$.
4. Find the sum and the product of the roots in $7x^2 + 3x - 2 = 0$.
5. Find a so that $3x - y = a$ is tangent to $x^2 + y^2 = 36$.
6. If $y = kx$, and $8 = k \cdot 3$, find y when $x = 15$.

CUMULATIVE REVIEW XXVIII

(After p. 196)

1. $\dfrac{a - 7}{a + 4} \div \left(\dfrac{a^2 + 6a + 8}{a^2 - 8a + 7} \cdot \dfrac{a^2 + 2a - 3}{a^2 + 2a - 8}\right)$.
2. Solve $\dfrac{1}{x} + \dfrac{1}{y} = 6$, $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 20$.
3. If $y = k/x$ and $12 = k/7$, find y when $x = 35$.
4. By using logarithms, evaluate $\dfrac{3.1 \times 4.2}{5.6}$.
5. Find the sum of the series 5, 10, . . . , 100.
6. Solve $x^2 + y^2 + 3x + 3y = 28$, $xy = 6$.

CUMULATIVE REVIEW XXIX

(After p. 209)

1. Find the sum of 2, 1, ½, to 8 terms.
2. Find the sum of the series 10, 20, . . . , 200.
3. Using logarithms, evaluate $\dfrac{42 \times 7.8 \times 3.5}{4.7 \times 81.4}$.
4. If $y = k/x^2$ and $100 = k/16$, find y when $x = 20$.

5. Solve $x^2 + 3y^2 = 57$, $3x^2 - y^2 = 11$.
6. Find a so that $ax - y = 6$ is tangent to $y = x^2 - 5x + 2$.

CUMULATIVE REVIEW XXX

(After p. 215)

1. The first term of an arithmetic series is 5, the last term is 57, and the number of terms is 14. Find the common difference.
2. Find the sum of 2, −1, ½, . . . to 8 terms.
3. Find the compound amount of \$1000 at 4% for 3 years.
4. $\dfrac{a^2 + 4a + 4 - b^2}{3a^2 - 7a - 6} \cdot \dfrac{3a + 2}{a + 2 + b} \cdot \dfrac{a - 3}{a + 2 - b}$.
5. If $z = kxy$ and $16 = k \cdot 2 \cdot 4$, find z when $x = 12$, $y = 16$.
6. $\dfrac{x^4 + x^2y^2 + y^4}{x^3 + y^3} \div \dfrac{x^3 - y^3}{x^2 - y^2}$, $\left(a^2 - \dfrac{b^6}{a^4}\right) \div \left(b - \dfrac{a^6}{b^5}\right)$.

CUMULATIVE REVIEW XXXI

(After p. 220)

1. Find the product $\dfrac{-1 + \sqrt{3}i}{1 - \sqrt{3}i} \cdot \dfrac{-1 + \sqrt{3}i}{1 + \sqrt{3}i}$.
2. Find by constructing graphs the approximate solutions of $y = x^2 - 6x + 4$, $3x - y = 3$.
3. Find a so that $2x - ay = 4$ is tangent to $y = x^2 - 7x + 4$.
4. Solve $x^3 + y^3 = 7$, $x + y = 1$.
5. Using logarithms evaluate $\sqrt{\dfrac{13.4 \times 2.9}{5.8 \times 1.7}}$.
6. In $S = \dfrac{n}{2}(a + l)$ find a if $S = 24$, $l = 14$, and $n = 8$.

CUMULATIVE REVIEW XXXII

(After p. 226)

1. For what value of x is $x^2 - 7x + 2$ greater than 0?
2. Find the present value of \$1000 due in 5 years, interest at 3%.
3. Find the amount of an annuity of \$850 for 8 years, interest at 3%.
4. Find the compound amount of \$2500 at 2% for 6 years.
5. Using logarithms evaluate $\dfrac{\sqrt[3]{2.8 \times 5.3}}{\sqrt{15.3 \times 27.5}}$.
6. Solve $S = \dfrac{n}{2}(a + l)$ for n, a, and for l in terms of the other letters.

CUMULATIVE REVIEW XXXIII

(After p. 235)

1. Transform $x^3 - x^2 + 5x - 8 = 0$ so as to decrease its roots by 1.
2. Locate real roots of $x^3 - 5x^2 + 7x - 4 = 0$.
3. Divide $x^4 - 3x^2 + 5x - 5$ by $x - 2$.
4. Find the present value of \$5000 due in 10 years, interest at 3%.
5. Insert 4 geometric means between 1 and 32.
6. If $a = 2$, $l = 78$ and $n = 13$, find d in $l = a + (n - 1)d$.

CUMULATIVE REVIEW XXXIV

(After p. 265)

1. $\dfrac{m^2 - n^2}{m^2 + n^2} \cdot \dfrac{m^4 - n^4}{m^3 - n^3} \div \dfrac{2}{m^2 + mn + n^2}$.
2. Using logarithms evaluate $(1.05)^7$.
3. Solve $x^3 - y^3 = 61$, $x - y = 1$.
4. Insert 6 geometric means between 2 and 3.
5. For what values of x is $x^2 + 2x - 9$ less than 0?
6. Locate real roots of $x^2 + 5x^2 - 7x + 4 = 0$.

CUMULATIVE REVIEW XXXV

(After p. 245)

1. In $x^3 - 5x^2 + 5x - 15 = 0$ what is the sum of the roots? their product?
2. Transform $x^3 - 9x^2 - 2x + 15 = 0$ so as to change the signs of its roots.
3. Transform $x^3 + 5x^2 - 9$ so as to decrease its roots by 2.
4. Divide $x^5 - 3x^3 + 6x - 4$ by $x + 2$.
5. For what values of x is $x^2 - 9x + 3$ less than 2?
6. $\left(\dfrac{1}{x + 2} + \dfrac{3}{x - 4}\right) \cdot \dfrac{x^2 - x - 12}{4x - 6}$.

CUMULATIVE REVIEW XXXVI

(After p. 245)

1. Approximate to three decimals the greatest root of $x^3 - 8x^2 - 4x - 2 = 0$.
2. In $x^3 - 8x + 16 = 0$, what is the product of the roots? What is the sum of the roots?
3. Find a maximum and a minimum of $x^3 - 5x^2 + 2x - 7$.
4. It is given that $x = \dfrac{3 + \sqrt{23}i}{2}$ is a root of $x^4 - 3x^3 + 11x^2 - 9x + 24 = 0$. Solve the equation completely.

5. Transform $x^4 - 5x^2 + 2x - 7 = 0$ so as to change the signs of its roots.
6. Find the amount of an annuity of \$500 for 6 years, interest at 3%.

CUMULATIVE REVIEW XXXVII

(After p. 274)

1. Find the sum and the product of the roots in $5x^2 - 9x + 4 = 0$, $9x - 5x^2 - 4 = 0$.
2. Find a so that $x + ay = 8$ is tangent to $x^2 + y^2 = 9$.
3. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{3}{4}$, find the value of $\sqrt{\frac{2a^2}{2b^2} + \frac{3c^2}{3d^2} + \frac{5e^2}{5f^2}}$.
4. Find the arithmetic, geometric, and harmonic means between 2 and 3. Reduce to decimals correct to 3 places. Compare.
5. Find the amount of an annuity of \$1200 for 20 years, interest at 3%.
6. Transform $x^4 - 9x^3 + 15 = 0$ so as to decrease its roots by 3.

CUMULATIVE REVIEW XXXVIII

 (After p. 303)

1. Using logarithms evaluate $\frac{1}{(1.04)^6}$.
2. Insert 8 geometric means between 1 and 6.
3. Transform $x^4 + 5x^3 - 2x + 8 = 0$ so as to change the signs of its roots.
4. In $x^3 + 5x^2 - 42x - 18 = 0$, what is the sum of the products of the roots taken two at a time?
5. Evaluate $\dfrac{\frac{a+b}{a-b} + 1}{\frac{a-b}{a+b} - 1}$ if $a = \frac{1}{3}$, $b = \frac{1}{4}$.
6. Five men and five women are placed around a table so that no two men are seated together. In how many ways can they be seated?

CUMULATIVE REVIEW XXXIX

(After p. 303)

1. Evaluate ${}_8P_4$, ${}_{10}P_8$, ${}_{12}P_{10}$.
2. Transform $x^5 + 4x^4 - 9x + 12 = 0$ so as to change the signs of its roots.
3. It is given that $-1 + i$ is a root of $x^4 + 4 = 0$. Find the other roots of this equation.
4. Using logarithms solve $5 = 3^x$.
5. Divide $x^4 + 2x^3 - 9x + 5$ by $x + 1$.
6. Transform $x^3 + 8x^2 - 10 = 0$ so as to decrease its roots by 1.

CUMULATIVE REVIEW XL

(After p. 303)

1. Find a maximum and a minimum of $x^3 + 7x^2 - 2x - 4$.
2. Find the sum and the product of the roots in $7x - 4x^2 + 8 = 0$, $-7x + 4x^2 + 8 = 0$.
3. Factor $x^2 - 9x + 3$, $x^2 + 9x - 3$.
4. Find the compound amount of \$7800 at 1% for 12 years.
5. Transform $x^4 + 6x^2 - 32 = 0$ so as to decrease its roots by 2.
6. $\left(\frac{1}{a+2} + \frac{2}{(a-2)^2}\right) \cdot \left(\frac{3}{a-2} - \frac{4}{(a+2)^2}\right)$.

CUMULATIVE REVIEW XLI

(After p. 303)

1. Factor $7x^2 - 3x + 4$, $2x^2 + 8x - 3$.
2. Using logarithms, solve $2 = (1.04)^x$.
3. Find the amount of an annuity of \$1500 for 15 years, interest at 3%.
4. Find the present value of \$2500 due in 15 years, interest at 3%.
5. Evaluate $\frac{{}_{12}P_{12}}{{}_{8}P_{6} \cdot {}_{10}P_{8}}$.
6. Find a maximum and a minimum of $x^3 - 9x^2 + 5x - 9$.

CUMULATIVE REVIEW XLII

(After p. 303)

1. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{2}{3}$, find the value of $\left(\frac{pa^3}{pb^3} + \frac{qc^3}{qd^3} + \frac{re^3}{rf^3}\right)^{1/3}$.
2. Find the compound amount of \$250 at $1\frac{1}{2}$% for 12 periods.
3. Divide $x^4 + 5x^3 - 6x^2 - 9x + 8$ by $x - 1$.
4. Transform $x^5 + 7x^3 - 29x + 5 = 0$ so as to change the sign of its roots.
5. Approximate to three decimals the greatest root of $x^3 + 9x^2 - 15 = 0$.
6. Without solving, determine the character of the roots of $5x^2 - 4x + 8 = 0$, $5x^2 + 4x - 8 = 0$.

CUMULATIVE REVIEW XLIII

(After p. 303)

1. Using logarithms, solve $3 = (1.05)^x$.
2. For what value of x is $x^2 - 6x + 2$ less than 6?
3. Divide $x^5 - 4x^4 + 5x^2 - 9$ by $x - 1$.
4. Eight red books and 6 blue books are placed in a row so that all blue books are together. In how many ways can this be done?

5. Using logarithms, solve $7 = (3.2)^x$.

6. Without solving, determine the character of the roots of $3x^2 - 9x + 1 = 0$, $3x^2 + 9x - 1 = 0$.

CUMULATIVE REVIEW XLIV

(After p. 303)

1. Find the compound amount of $380 at 2% for 10 periods.

2. Find the present value of an annuity of $1200 for 10 years, interest at 3%.

3. Locate the real roots of $x^4 - 8x^2 - 7x + 5 = 0$.

4. In how many ways can a committee of 5 be selected from a group of 12?

5. Find a maximum and a minimum of $x^3 - 12x + 8$.

6. Approximate to three decimals the greatest root of $x^4 - 8x^2 + 3x - 10 = 0$.

CUMULATIVE REVIEW XLV

(After p. 303)

1. Find the amount of an annuity of $600 for 25 years, interest at 3%.

2. For what value of x is $x^2 + 5x + 8$ greater than 10?

3. Locate the real roots of $x^5 - 3x^3 + 5x - 7 = 0$.

4. In how many ways can a committee of 3 women and 2 men be selected from a group of 7 women and 5 men?

5. Approximate to three decimals the greatest root of $x^4 - 10x^3 + 8x - 20 = 0$.

6. Evaluate ${}_5C_3$, ${}_7C_5$, ${}_{12}C_{10}$.

CUMULATIVE REVIEW XLVI

(After p. 303)

1. Find the present value of an annuity of $1500 for 20 years, interest at 3%.

2. Locate the real roots of $x^5 + 3x^3 - 5x + 7 = 0$.

3. It is given that $x = 1 + \sqrt{3}\, i$ is a root of $x^4 + 2x^2 + 4x + 8 = 0$. Find the other roots of this equation.

4. On a shelf there are 12 red books, 9 green books, and 7 blue books. In how many ways can 8 red books, 6 green books, and 3 blue books be taken from the shelf?

5. In a bag there are 7 white balls and 5 black balls. In taking 4 balls what is the probability that 2 will be white and 2 black?

6. Approximate to three decimals the greatest root of $x^3 - 25x + 3 = 0$.

CHAPTER 27:

HISTORICAL SKETCH

The principal sources used in compiling this historical sketch are:

1. Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, fourth edition, 1906.
2. Florian Cajori, *A History of Mathematics*, second edition, 1924.
3. David Eugene Smith, *History of Mathematics*.
4. W. W. Rouse Ball, *A Short Account of the History of Mathematics*, 1893.
5. J. Gow, *History of Greek Mathematics*, 1884.

Use has also been made of Heath's *The Thirteen Books of Euclid's Elements* and other standard works.

Smith has been followed in dates and spelling of names.[1]

Sources of quotations are given in the text.

After studying algebra in our secondary schools, a bright pupil is able to solve the equation $ax^2 + bx + c = 0$, and to render an intelligent account of what he is doing. Simple as this operation now seems to us, some of the elements involved in it were finally discovered and perfected thousands of years after algebraic processes were first used. The early part of this sketch is largely concerned with the story of how mankind learned to solve this equation, and the first-degree equation which is subsidiary to it. An analysis of this process into its various elements seems pedantic, but historically each of the items presently to be enumerated was of real importance; each of them offered, so to speak, stubborn and often prolonged resistance to man's effort toward its discovery.

The following classification of the elements involved will serve our purpose.

1. A system of numbers had to be invented and generalized so as to include the ordinary numbers of arithmetic (integers and rational fractions), irrational numbers such as $\sqrt{2}$, directed or signed numbers, and the complex number.

[1] These sources are all in English except the large work of Cantor. The student will find the works by Ball, Cajori, and Smith very readable. Smith's *History* contains many odd and interesting items.

2. In order to make numerical computation reasonably convenient it was necessary to have a system of numerals such as our present system, including its extension to include decimal fractions.

3. A system of algebraic symbols indicating operations to be performed had to be perfected. This includes the signs $+$, $-$, $\times$, $\div$, $=$, $\sqrt{\ }$, the meaning of exponents, and other symbols.

4. The use of symbols other than numerals (letters for instance) to represent numbers had to be developed. This includes the use of letters to represent unknown numbers for whose values search is being made; and also literal coefficients and exponents, so that a single solution may yield a formula in which any particular values may be substituted.

5. The use of such general symbols (letters) to represent numbers made it necessary to perfect rules for operating on them. That is, the rules for performing the operations of arithmetic had to be adapted so as to be applicable to this new symbolism.

6. It was necessary to perfect a set of principles governing the operations which may be performed upon an equation in the process of solving it.

The complete development of each of these elements extended over a long period of time and contributions were made by many national and social groups, often inhabiting widely separated geographical regions. Cumbersome and inadequate symbolisms and methods were brought to a high degree of refinement and complexity, to be discarded completely upon the advent of new discoveries and inventions. Communication of information was almost unbelievably slow; simple and extraordinarily useful discoveries required a thousand years to find their way from India to England. Discoveries did not always come in the order of simplicity when simplicity is measured by the ease of understanding once the discovery had been made. Men of the highest genius solved problems which by the methods they used were as difficult as any that have ever been solved, while at the same time they failed utterly to hit upon apparently obvious methods that would have reduced their tasks to schoolboy problems. Many a time in the story of our own country did an eager prospector pass unwittingly places where later the richest of mines were found. So it has been with those venturesome intellectual spirits who have searched for treasure that when found contributed to the enrichment of us all. Does not this make us wonder whether there may not now lie within easy reach new ways of thinking and doing that are not to be discovered for another thousand years, which when once found will again reduce some of our most difficult problems to mere child's play? One reason for answering this question in the negative may be that never has there been such extensive, systematic, and unprejudiced search for truth as there is at present. For this reason it is likely that the simpler and more obvious discoveries have already been made.

Babylonians, Egyptians, Greeks

The number system.—By far the most generally used system of numbers is the decimal system which is built on the base of ten. This is so among the peoples now inhabiting the earth, and also among those who have left historical records. Curiously enough, one of the earliest civilizations of which we have definite record forms an exception to this rule, and equally curious is it that this exception is of real interest to us, inasmuch as it gave rise to certain usages that are now practically universal. The Babylonians as early as 3000 B.C. used 60 as a base (the sexagesimal system). It is surmised that two peoples, one having a decimal system and one a system with six as the base, formed a mixed Babylonian population and that the sexagesimal system was a result of this mixture ($6 \times 10 = 60$). Certain it is that the base 10 was used along with the base 60. There is some reason for believing that the Babylonians divided the day into 60 parts (hours) and it is known that such a division has been used in India. It is to be noted that the Babylonians used the sexagesimal system consistently also for fractions as we now use the decimal system. We do not know why the Babylonians divided the circle into 360 degrees, but with the sexagesimal system in common use it was natural to divide the degree, the hour, and the minute into 60 parts.

The Egyptians and the Greeks used the decimal system for integers but not for fractions. The very considerable difficulty which earlier peoples had in dealing with fractions was met in two ways. The large denominator 60 enabled the Babylonians to express all fractions in practical use with the same denominator: thus $\frac{1}{2}$ and $\frac{1}{3}$ were expressed as $\frac{30}{60}$ and $\frac{20}{60}$. The Romans used the denominator 12, even to the extent of writing $\frac{1\frac{1}{2}}{12}$ instead of $\frac{1}{8}$. The Egyptians, and to a large extent the Greeks, used only unit fractions. For instance, instead of $\frac{2}{5}$ they used $\frac{1}{3} + \frac{1}{15}$. It was not until late in Greek history that fractions came to be regarded as numbers. Earlier they were "ratios" or indicated problems in division. Unity, 1, was regarded not as a number, but as the "source and origin of all numbers." There is no trace of signed or directed numbers, and irrational quantities were not regarded as numbers.

Systems of Notation; Computation.—The distinctive characteristic of our system of numerals is the principle of place value by which a digit may represent units or tens or millions, depending upon the place it occupies. To make such a system effective it is necessary to have a symbol zero to indicate the absence of quantity. The Babylonians had a complete system of this kind certainly as early as 300 B.C. Astronomical writings of this time contain fractions equivalent to $\frac{4}{60}\quad\frac{11}{60^3}\;\frac{32}{60^4}$, juxtaposition indicating addition. Only the numerators are written, the order indicating the denominators. In this number the denominator 60^2 is lacking and this is indicated by leaving an empty space. At times a symbol ⋞ is written in this space and is therefore made to serve exactly the same purpose as our zero. While the sexagesimal system was in part imported into Greece, this excellent system of notation for numbers remained entirely unknown to western civilization for another fifteen hundred years.

The later Greek notation for numbers was similar to that of the Romans. To the twenty-four letters of the Greek alphabet were added three symbols, making twenty-seven in all. The first nine of these were made to represent the numbers 1, 2, . . . , 9; the next nine to represent 10, 20, . . . , 90; and the last nine to represent 100, 200, . . . , 900. This made it possible to write any number up to 999. Larger numbers were written by means of special devices which we shall not describe. Fractions were written by placing the numerator first and then the denominator, the special meaning of the symbols being indicated by means of accents, double accents, and a repetition of the symbols indicating denominators. Written symbols to indicate numbers were in general use long before these symbols were used in computing. In fact, symbols such as those used by the Greeks could not easily be used for this purpose. "Finger reckoning" and an instrument called the abacus were used almost universally, and the abacus (called by the Chinese the swan pan) is still in use among the Chinese and Japanese. These were used even by the Babylonians whose system of numerals was the equivalent of our modern system and could easily be used directly in computing. There is no indication that the Babylonians used their symbol for zero in computing.

The Ahmes papyrus.—The beginnings of all history reveal civilizations already in existence. There are records of Egyptian dynasties as far back as 4500 B.C.; pyramids erected 3700 B.C. are perfectly oriented and their sides slope at a very nearly uniform angle of 52°, showing that elaborate measurements had been made. From a period 2000 years later (about 1700 B.C.) we have the oldest mathematical book known; it is supposed to be a reproduction, in part at least, of older manuscripts dating from before 3000 B.C. This manuscript, which was written by one Ahmes, is now in the Rhind collection in the British Museum, and is often called the Rhind papyrus. It was written for the expert mathematicians of that time, and elementary matter—treatment of integers, for instance—is omitted.

The first part of this book deals with fractions. The object is to reduce fractions to unit fractions and the work is confined to fractions whose numerator is 2. Thus $\frac{2}{9}$ is reduced to $\frac{1}{6}+\frac{1}{18}$. The problem is stated "express 2 divided by 9." Hence it seems likely that what we now represent by $\frac{2}{9}$ arose in connection with division and was not regarded as a number. A table is given of the reduction of all fractions of the form $\frac{2}{2n+1}$ up to $\frac{2}{99}$. There is no clue to the method used in reducing fractions to unit fractions and it is not apparent that any particular rule was used. The general conclusion of Gow (and also of Cantor) is: "We must conclude that the table was compiled empirically by different persons and at different times, and certainly without any general theory."

This method, or an analogous one, of dealing with fractions was very widespread in antiquity. To quote Gow:

> But division, when it came to be conducted with nicety, introduced a new difficulty. The divisor was not always a whole factor of the dividend and there was then a remainder. What was to be done with this? The question, no doubt, first arose with concrete units, in a case, for instance, where 23 apples were left to be divided among 24 men. Here obviously each man will get a fraction of an apple, but there are two ways of ascertaining the fraction. One is to divide each apple into 24 equal parts, and to give to each man 23 such parts. The other is to subdivide 23 into groups, say 12, 8 and 3, and so to give each man first $\frac{1}{2}$, then $\frac{1}{3}$rd, then $\frac{1}{8}$th of an apple. This latter method of treating a remainder (by taking parts of it at a time) is clearly analogous to the way in which the whole dividend has previously been treated, and no doubt it recommended itself, on this account, to the calculators of antiquity.

After this preliminary work on fractions Ahmes proceeds to the solution of problems which we now solve by means of first degree equations in one unknown. The unknown is designated by the word "hau" or "heap." We have (Cantor):

"Heap, its seventh, its whole makes 19."

$$\text{i.e.,} \frac{x}{7} + x = 19.$$

"Heap, its $\frac{2}{3}$, less $\frac{1}{3}$ [of result] leaves 10."

$$\text{i.e.,} (x + \frac{2}{3}x) - \frac{1}{3}(x + \frac{2}{3}x) = 10.$$

"Heap, its $\frac{2}{3}$, add $\frac{1}{3}$ [of result], subtract $\frac{2}{3}$ [of last result] leaves 10."

$$\text{i.e., } x + \frac{2}{3}x + \frac{1}{3}(x + \frac{2}{3}x) - \frac{2}{3}[(x + \frac{2}{3}x) + \frac{1}{3}(x + \frac{2}{3}x)] = 10.$$

"Heap, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole make 33."

$$\text{i.e., } \frac{2}{3}x + \frac{x}{2} + \frac{x}{7} + x = 33.$$

The solution of this last equation was essentially as follows. Suppose x (heap) is 1. Then the first member (in Egyptian) is $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}$, the juxtaposition indicating addition. The problem is then to find the number by which $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}$ must be multiplied to make 33. This is done by successive "trial" multiplications, and the result is $14\ \frac{1}{4}\ \frac{1}{97}\ \frac{1}{56}\ \frac{1}{679}\ \frac{1}{776}\ \frac{1}{194}\ \frac{1}{388}$. The table mentioned above gives $\frac{2}{97} = \frac{1}{56}\ \frac{1}{679}\ \frac{1}{776}$ and would be of value in finding this result.

By our method the equation at once reduces to $97x = 1386$, and $x = 14\ \frac{28}{97}$. The actual work of reducing $\frac{28}{97}$ to $\frac{1}{4} + \frac{1}{97} + \frac{1}{56} + \frac{1}{679} + \frac{1}{776} + \frac{1}{194} + \frac{1}{388}$ gives some idea of the amount of trouble avoided by using methods developed since Ahmes.

The Egyptians were evidently acquainted with arithmetic series and possibly with geometric series. Thus we have the following problem and solution (Gow).

"Ten measures of corn for 10 persons. The difference between each person's share and the next's is $\frac{1}{8}$th of a measure." The solution runs: "I find the mean, one measure. Take 1 from 10: remainder 9. Halve the difference, i.e., $\frac{1}{16}$. Take it 9 times, that gives you $\frac{1}{2}\ \frac{1}{16}$. Add it to the mean. Deduct $\frac{1}{8}$th of a measure for each person so as to reach the end." These consecutive sentences mean, in modern algebraical form: Find $\frac{s}{n}$. Find $(n - 1)$. Find $\frac{b}{2}$. Find $\frac{b}{2} \times (n - 1)$. Add $\frac{s}{n}$ to $\frac{b}{2}(n - 1)$, i.e., these directions imply a knowledge of the formulae for finding the sum or the first term of an arithmetical progression.

To quote from Cajori:

> That the period of Ahmes was a flowering time for Egyptian mathematics appears from the fact that there exist other papyri (more recently discovered) of the same period. They were found at Kahun, south of the pyramid of Illahun. These documents bear close resemblance to Ahmes. They contain, moreover, examples of quadratic equations, the earliest of which we have a record. One of them is: A given surface of, say, 100 units of area, shall be represented as the sum of two squares, whose sides are to each other as $1:\frac{3}{4}$. In modern symbols, the problem is, to find x and y, such that $x^2 + y^2 = 100$ and $x:y = 1:\frac{3}{4}$. The solution rests upon the method of false position. Try $x = 1$ and $y = \frac{3}{4}$, then $x^2 + y^2 = \frac{25}{16}$ and $\sqrt{\frac{25}{16}} = \frac{5}{4}$. But $\sqrt{100} = 10$ and $10 \div \frac{5}{4} = 8$. The rest of the solution cannot be made out, but probably was $x = 8 \times 1, y = 8 \times \frac{3}{4} = 6$.

Greek contributions to algebra.—No attempt will be made in this sketch to trace the chronological development of algebra in Greece, but rather to indicate the sum total of Greek contribution. The story begins with Pythagoras 569–500 B.C. and ends with Diophantus who lived about A.D. 250.

In dealing with fractions the Greeks did little in advance of the ancient Egyptians as represented by Ahmes. While they had a symbolism by means of which any fraction could be represented, and while some of their mathematicians, such as Diophantus, made actual use of it, for practical purposes they continued till the end to use the ancient devices. An exception in the case of "astronomical fractions" has already been noted. Sometimes a fraction was reduced approximately to unit fractions. Thus Eutocius, in the 6th century after Christ, reduces $\frac{15}{64}$ to $\frac{1}{6} + \frac{1}{15}$ which is $\frac{1}{960}$ too small.

Though the Greeks were thoroughly acquainted with incommensurable quantities, they had no arithmetic symbol to represent even the simplest cases of incommensurable numbers—not even $\sqrt{2}$. In fact, it is very doubtful whether they regarded the length of the diagonal of a unit square as representable by a number. The more one studies the work of the Greeks within the field which we now call algebra, the more evident does it become that they confined themselves to the ordinary numbers of elementary arithmetic. They had no negative numbers and no irrational numbers. By far the highest point in Greek algebra was reached by Diophantus, who rejected all negative or irrational solutions of equations; such solutions were "impossible" or "absurd."

If Diophantus were to solve the equation $x^2 - 8x + 12 = 0$, he would proceed in substance as follows.

$$\begin{aligned} x^2 - 8x + 12 &= 0 \\ x^2 - 8x + 16 - 4 &= 0 \\ x^2 - 8x + 16 &= 4 \\ x - 4 &= 2 \\ x &= 6. \end{aligned}$$

That is, with him 4 would have only one square root and hence he would find only one root, 6, of the given equation, though it has another positive root, 2. Diophantus could find the product $(x-1)(x-2)$ provided x is greater than 2. In this connection he states the rule "a subtraction times a subtraction is an addition," but he did not have any conception of a negative number.

Gow says:

Euclid (300 B.C.) gives many propositions which, though they profess to deal with geometrical magnitudes, are at once applicable to numbers, and may have been so intended. Stated in modern symbolism some of these are:

(1) $ab + ac + ad + \dots = a(b + c + d + \dots)$

(2) $(a + b)^2 = (a + b)a + (a + b)b$

(3) $(a + b)a = ab + a^2$

(4) $(a + b)^2 = a^2 + b^2 + 2ab$

(5) $\left(\frac{a}{2}\right)^2 = (a - b)b + \left(\frac{a}{2} - b\right)^2$

(6) $(a + b)b + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2} + b\right)^2$

(7) $(a + b)^2 + a^2 = 2(a + b)a + b^2$

(8) $4(a + b)a + b^2 = (2a + b)^2$

(9) $(a - b)^2 + b^2 = 2\left(\frac{a}{2}\right)^2 + 2\left(\frac{a}{2} - b\right)^2$

(10) $b^2 + (a + b)^2 = 2\left(\frac{a}{2}\right)^2 + 2\left(\frac{a}{2} + b\right)^2$

There are geometric constructions equivalent to solving the equations $x(x-a)=a^2$, $x^2=bc$. Limitation of space makes it impossible to give an account of Euclid's treatment of surds (square roots of magnitudes). Never did he treat of algebraic operations as such; he used no algebraic notation or machinery. He proved geometric theorems and solved problems not by the use of equations but by making constructions with ruler and compasses. Nevertheless, his results could not fail to have algebraic meaning to those who later turned their thoughts in this direction.

By the time of Diophantus (five and a half centuries had intervened) a definite change had taken place. Mathematicians now occupied themselves unequivocally with algebraic problems and processes. Diophantus solved equations like $84x^2+7x=7$ and $630x^2+73x=6$, though he does not clearly describe his process. But he gives the "answer," the one answer which, after "completing the square" in the left member, is found by taking the positive square root of the right member. Three types of quadratics are distinguished. In modern notation these are $ax^2+bx=c$, $ax^2+c=bx$, $ax^2=bx+c$. There was not the faintest idea that the letters might be negative and that all these types could be represented by $ax^2+bx+c=0$.

Heron of Alexandria (about 100 B.C.), who is supposed to have been an Egyptian, though certainly learned in Greek mathematics, was perhaps the most daring in his dealing with the equation. To quote Gow once more:

Algebra would come only from the practical calculator who was not hampered by such difficulties. [Euclid would not add a line to a square or divide a line by another line; the ratio of two lines was to him a different matter, not a number.] The first step seems to have been taken, not by a Greek, but by the Egyptian, Heron. Thus in a proposition . . . he does not scruple to add an area to a circumference. In modern symbols, the proposition runs: "If S be the sum of the area (A), the circumference (C), and the diameter (d), of a circle, find the diameter." The answer which he gives is $d=\dfrac{\sqrt{154S+841}-29}{11}$. The proof, which he does not give, is obviously as follows: A is $\frac{\pi}{4}d^2$: C is πd: π is $\frac{22}{7}$. Then $S=\frac{\pi}{4}d^2+(\pi+1)d=\frac{11}{14}d^2+\frac{29}{7}d$. Multiply each term by 154 ($=11\times 14$). Then

$121d^2 + 638d + 841 = 154S + 841$ or $(11d + 29)^2 = 154S + 841$, from which Heron's answer immediately follows. It cannot be doubted that Heron could solve an impure quadratic equation in a way which, but for the want of a symbolism, would be simply algebraical.

The Greeks made some advances in algebraic symbolism. Diophantus used special symbols for the unknown in a problem and for powers of the unknown, but these symbols are not used to indicate powers of any other number. Addition is indicated by juxtaposition; the coefficient of the unknown is written after it and in this case juxtaposition indicates multiplication. Subtraction and equality are designated by special symbols. Numbers to be subtracted are always written after all the others.

In a way, by the time of Diophantus, algebra was, so to speak, in the air. We find puzzles proposed for popular amusement which naturally lead to algebraic solutions. Some of these are known to every pupil who studies algebra.

Of four pipes, one fills a cistern in one day, the next in two days, the third in three days, the fourth in four days: if all run together, how soon will they fill the cistern?

The mule says to the donkey: If you gave me one measure I should carry twice as much as you: if I gave you one, we should both carry equal burdens. Tell me their burdens.

Demochares has lived $\frac{1}{4}$th of his life as a boy, $\frac{1}{5}$th as a young man, $\frac{1}{3}$rd as a man, and 13 years as an old man. How old is he?

The age of Diophantus may be computed from the epitaph:

Diophantus passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor; five years after his marriage was born a son who died four years before his father, at half his father's age.

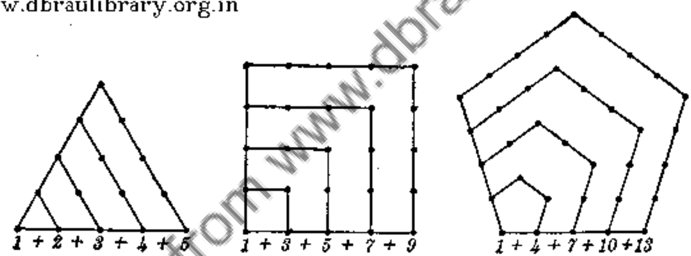
The Greeks made many minor and incidental discoveries in the general field of algebra, some of which now appear as exercises and problems in books like the present. Pythagoras found integers which satisfy the equation $a^2 + b^2 = c^2$. That is, he found sets of three integers which represent the lengths of the sides of a right triangle. The Egyptians knew that 3, 4, 5 are the sides of such a triangle. Pythagoras discovered that for all integral values of n, $2n+1$ and $2n^2+2n$ are the sides of a right triangle of which $2n^2+2n+1$ is the hypotenuse. That is,

$$\begin{aligned}(2n+1)^2+(2n^2+2n)^2 &= 4n^4+8n^3+8n^2+4n+1\\ &= (2n^2+2n+1)^2.\end{aligned}$$

Plato discovered that for all integral values of n, $2n$ and n^2-1 are the sides of a right triangle whose hypotenuse is n^2+1. That is,

$$(2n)^2+(n^2-1)^2 = n^4+2n^2+1 = (n^2+1)^2.$$

Much attention was given to what were called polygonal numbers. Consider the series $1+2+3+\ldots+n=\frac{n}{2}(n+1)$ (see page 298).



The terms of this series may be arranged in successive layers forming a triangle. Hence the sum $\frac{n}{2}(n+1)$ was called a triangular number. The terms of the series $1+3+5+7+\ldots+(2n-1)$ $=\frac{n}{2}(2n)=n^2$ is a square number since the successive terms may be arranged on two sides of a square so as to form a series of successive squares. The terms of the series $1+4+7+10+\ldots+(3n-2)$ $=\frac{n}{2}(3n-1)$ is a "pentagonal" number since the successive terms may be arranged so as to form a series of regular pentagons.

In this manner the sum of the general series

$$1 + (1 + k) + (1 + 2k) + \ldots + 1 + (n-1)k$$
$$= \frac{n}{2}\Big[(n-1)k + 2\Big]$$

came to be called a polygonal number, the number of sides being $k + 2$.

In a similar manner "polyhedral" numbers were constructed. There were pyramidal numbers, the simplest of which is the tetrahedral number, represented by the number of shots in a triangular pile.

Many peculiar properties of numbers were discovered. It was noticed that the cubes of integers may be found by adding odd numbers. That is,

$$1^3 = 1,\ 2^3 = 3 + 5,\ 3^3 = 7 + 9 + 11,$$
$$4^3 = 13 + 15 + 17 + 19, \ldots$$

The differences of consecutive squares give the odd numbers. That is, $4 - 1 = 3,\ 9 - 4 = 5,\ 16 - 9 = 7,\ 25 - 16 = 9, \ldots$ Again, with modern notation, $n(n + 1) - n(n - 1)$ taken for consecutive values of n gives the even numbers.

With our notation and methods these are all very simple, but to the Greeks they appeared as remarkable discoveries. There was not a little mysticism connected with it all. The Pythagoreans noticed that certain simple numerical relations of the lengths of vibrating strings regulated the terms in the various parts of the musical scale, and they began to talk wildly about the music of the spheres, and about numbers being the key to all mysteries. To quote Gow:

> It was natural to a nascent philosophy to draw, by false analogies and the use of a brief and deceptive vocabulary, enormous conclusions from a very few observed facts: and it is not surprising if Pythagoras, having learnt in Egypt that number was essential to the exact description of forms and of the relations of forms, concluded that number was the cause of form and so of every other quality. Number, he inferred, is quantity and quantity is form and form is quality.
>
> Primitive men, on seeing a new thing, look out especially for some resemblance in it to known things, so that they may call both by the same name. This develops a habit of pressing small and partial analogies. It also causes many meanings to be attached to the same word. Hasty and confused theories are the inevitable result.

Still another kind of problem was considered by Diophantus, the last and the greatest of the Greek algebraists. The following are typical examples:

To find three numbers, such that the square of each plus the next number is a square. To find three numbers, such that their sum and also the sum of any two of them shall be a square. To divide a given number into four parts, such that the sum of any three parts is a square. Gow remarks:

> Problems of this sort should be capable of general solutions. They are intended to discover classes of numbers having a common property. . . . But Diophantus does not, in fact, treat them generally. He is satisfied with a solution which gives only one case or a few cases.[1] Usually he arrives at an equation to which he finds only one particular solution. Even where the problem leads to a quadratic equation, which may be solved for two positive roots, he never gives more than one. . . . It must be added also that he will not accept a result which is either a negative or an irrational quantity. . . . On the other hand, he does not by any means object to a fractional result, and he is the first of the Greeks to whom a fraction was a number and not a ratio.

Solution of problems of this type has come to be called Diophantine analysis, and is not included in an elementary modern course on algebra such as the present. The Hindus, as we shall see later, treated such problems more generally and under more stringent conditions than did Diophantus, and the modern treatment might perhaps more appropriately be called Hindu analysis.

This brief account, too long in proportion in this short sketch, shows essentially what the Greeks did, though many details are omitted. No one doubts the transcendent intellectual powers of the Greeks, and, in view of their infinitely greater achievements in geometry, it is cause for wonder that their contribution to algebra was comparatively meager. Was this due to lack of a system of numerals by means of which computation would be possible? Was it perhaps due to the very high standard of logical excellence to which they brought geometry early in their history? To bring numbers and algebra to the same standard would require discoveries

[1] In this respect his work was less general than cases quoted above from Pythagoras and Plato.

which were finally made some seventy years ago. How clearly the Greeks dealt with irrational quantities in geometry in perfecting their method of exhaustion; and what a stumbling block the irrational number proved! It was not before about 1870 that this was finally cleared up. Can it be that the Greeks, seeing as they did that numbers and algebra could not be treated by them on the high plane of logical rigor which they had given to geometry, found the former repulsive? Did their very learning in geometry unfit them to be the creators of algebra? Is it more than coincidence that Heron, an Egyptian, was more emancipated than Diophantus? We can only conjecture.

Hindus, Arabs, Chinese

We shall begin with the Hindus. Some fourteen hundred years before Christ, peoples of the Aryan race invaded India from the northeast and gradually made themselves masters of the whole peninsula. A rigorous caste system developed in which the original population, crinkly haired Negritos mixed with other strains became the lower castes, laborers and servants, while the conquerors became the upper caste of priests and warriors—the Brahmins and the "Kshatriyas." These latter developed a culture in which the lower castes had no part and which was not in the least understood by them. Learning of all sorts came to be confined to the Brahmins whose written work at least was wholly in Sanscrit, the language of the invaders, which remained forever unknown to the lower castes. All the mathematical work of the Hindus was written in Sanscrit.

Aryabhata; Brahmagupta; Bhāskara.—It is necessary to mention at least three names in the history of Hindu mathematics. The year of the birth of each of these is definitely known: Aryabhata, 476 A.D.; Brahmagupta, 598 A.D.; and Bhāskara, 1114 A.D. Little that is definite is known about the course of development of Hindu mathematics before Aryabhata. The last-mentioned evidently knew a general rule for solving the quadratic, though it is not certain that he went beyond the Greeks. Brahmagupta marks the highest point in Hindu algebra, though minor improvements were made later. Bhāskara made certain advances, some of which are noted below.

What the Hindus learned from others.—It is certain that by the time of the first centuries of our era there was, and had been for some time, considerable communication between India, on the one hand, and Babylonia, China, Egypt, and Greece on the other. There is also definite evidence in the writings of the Hindus that they were indebted to Greece and China. There is little doubt but that the sexagesimal system of the Babylonians was carried to India. We know that they learned from the Chinese, for Brahmagupta repeats a problem which appears several centuries earlier in Chinese writings, and Greek expressions are used which show that the writings of the west were known to them, at least in part.

What the Hindus contributed.—We have seen that with some exceptions, the mathematical work of the Greeks was essentially geometrical. The work of the Hindus was essentially arithmetical and algebraic. Assuming that a system of notation involving the principle of place value was imported from Babylonia, they certainly perfected and adapted it to a uniform decimal system for integral numbers. But they failed to develop decimal fractions, using ordinary common fractions as we do. They also retained the Babylonian sexagesimal fractions for astronomical purposes. A set of nine digits was perfected and later a symbol for zero was introduced, making a complete system of notation. Their name for zero was "sunya," which meant "empty." By the end of the fourth century of our era the complete system of "Arabic numerals" was in common use among the learned Brahmins. Computing was done by them by means of these numerals; they did not use the abacus.

Before the end of the third century, probably much earlier, the Chinese had considered problems which led to equations like $x^2 + 34x - 35500 = 0$, but the rule for the solution is hazy and it cannot be said with any confidence that they possessed a general rule for solving such an equation. It is practically certain that the Hindus knew about such Chinese writings. It is further extremely likely that they learned about the work of Heron and Diophantus, and it is also certain that the work of these Greeks was at least clearer and more explicit than that of the Chinese. How, then, did the Hindus improve on the Greeks? In several important respects:

1. *Algebraic notation was improved.* Addition was indicated by juxtaposition as with Diophantus. Subtraction was indicated by placing a dot above the number to be subtracted; multiplication was indicated by writing *bha* after the numbers to be multiplied; division by writing the divisor below the dividend (omitting our horizontal stroke between them); and square root by *ka*. Their symbol for the unknown was *ya*, the initial letters of the word yavattavat which Cantor (and others) translates "as much as" (quantum tantum). If more than one unknown were involved, names of colors were used to indicate them. There was also a symbol indicating equality, but this was not needed, since the second member of an equation was always written below the first.

2. *The system of numbers was definitely extended to include negative numbers.* The favorite illustration was loss and gain, but they also used signed numbers to indicate opposite directions on a line as we do. Bhāskara says explicitly that the square of a positive number, and also of a negative number, is positive, and that a negative number has no square root. The sign for subtraction was also the sign of a negative number and it was thus fully recognized that the addition of a negative number is equivalent to the subtraction of a positive number. It followed that the three types of quadratic equations recognized by Diophantus were reduced to the single equation $ax^2 + bx + c = 0$.

3. *The irrational quantity was fully recognized as a number.* To quote Cajori:

> The Hindus never discerned the dividing line between numbers and magnitudes, set up by the Greeks, which, though the product of a scientific spirit, greatly retarded the progress of mathematics. They passed from magnitudes to numbers and from numbers to magnitudes without anticipating that gap which to a sharply discriminating mind exists between the continuous and discontinuous. Yet by doing so the Indians greatly aided the general progress of mathematics.

They operated freely with radicals. Thus Bhāskara uses the formula $\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$ to show that $\sqrt{10+\sqrt{24}+\sqrt{40}+\sqrt{60}} = \sqrt{2}+\sqrt{3}+\sqrt{5}$.

4. *Definite rules were formulated for solving the quadratic.* The method finally adopted for solving the quadratic was what is now popularly called the Hindu method. The steps are:
$ax^2 + bx = c$, $4a^2x^2 + 4abx = 4ac$, $4a^2x^2 + 4abx + b^2 = 4ac + b^2$,
$(2ax + b)^2 = 4ac + b^2$, $x = \dfrac{\sqrt{4ac + b^2} - b}{2a}$.
Note that only one sign of the radical is given. Bhāskara remarks that negatives are not "accepted by the people." The device of multiplying by $4a$ is attributed to one S'ridhara, who lived after Brahmagupta.

Miscellaneous contributions of the Hindus.—The Hindus used the binomial formula to find square and cube roots; they knew the sum of the arithmetic and geometric series and also of $1^2 + 2^2 + \ldots + n^2$ and of $1^3 + 2^3 + \ldots + n^3$; they rationalized the denominator in fractions and they used formulas for permutations and combinations. Finally, Bhāskara considered the third degree equation and gave a solution for the case when it could be reduced to the form $(x - a)^3 = b$.

The very important contributions of the Hindus to the solution of indeterminate equations will not be considered here except to remark that integral solutions were insisted upon and that general solutions were sought and in many cases found, thus showing a great advance over Diophantus in the so-called "Diophantine Analysis."

Though the use of zero was definitely brought into mathematics by the Hindus, they had rules for its use, some of them childish, some of them not so much so. Brahmagupta says: $\dfrac{0}{0} = 0$, $a \div 0 = \dfrac{a}{0}$, and Bhāskara: "$\dfrac{a}{0}$ is not changed by adding to it or subtracting from it." A commentator, Krishna, says: "The more the divisor is diminished the more is the quotient increased. If the divisor is decreased to the limit the quotient is increased to the limit. But so long as it (the quotient) can be stated so long it is not increased to the limit, for a larger number can be given. The quotient $(a \div 0)$ is therefore of indeterminate magnitude and is therefore properly said to be infinite." This last shows real insight, but see what they also did: Bhāskara does the equivalent of the following.

$$\frac{x}{0} + x - 9 = \frac{x}{0}; \left(\frac{x}{0} + x - 9\right)^2 + \frac{x}{0} = 90.$$

Then, $\frac{x^2}{0} + \frac{x}{0} = 90$, or $x^2 + x = 90$ and $x = 9$.

The Hindus wrote scientific work in poetic form and sometimes even the substance had a poetic flavor. This made it difficult to produce clear exposition. Here is one of their problems:

> The square root of half the number of bees in a swarm has flown out upon a jessamine bush, $\frac{8}{9}$ of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus flower into which he was allured in the night by its sweet odor, but is now imprisoned in it. Tell me, most enchanting lady, the number of bees.

But their poetically stated problems were not all addressed to most enchanting ladies:

> Of a herd of apes, the square of one eighth were hopping about in the trees enjoying themselves in play, the remaining 12 could be seen gossiping [not "chattering"] together on a hill. How many were there in the herd?

Thus wrote the Brahmins of India.

The Arabians.—In the short space of one hundred years, a few obscure tribes from the Arabian desert became the political masters of an empire as great in extent as that of Rome at its zenith. Arabia, Persia and Babylonia, all of Asia Minor, Egypt and the whole of Africa north of the Great Desert, and the Spanish peninsula, all came under their sway. When the flood of conquest subsided, these Mohammedan Arabs turned their energies to the arts of peace. Their first work in the service of science and scholarship was to translate into Arabic almost a complete set of scientific and philosophical writings of western and near eastern peoples. Native scholars of India, Persia, Mesopotamia, Egypt, and Greece assembled at Bagdad, the eastern of the two seats of Mohammedan empire, where they translated writings from their own languages into Arabic. It thus came about that in the ninth century the Arabs had at their disposal in their native tongue practically all the scientific work that had ever been written. No doubt the greatest value of this work was in the preserving for modern Europe, though at times in corrupt form, much that would otherwise have been lost permanently or brought to light much later than it was.

The contribution of the Arabs.—The total of Arabic contribution to arithmetic and algebra was not great. From among the varied symbols representing the integers 1 . . . 9 they selected what seemed the most convenient, and standardized the use of a definite set, including a symbol for zero. The Indian notation was in this way adopted in its totality. They also improved some of the forms of computation.

The name algebra derives from Arabic words. Early in the ninth century one Mohammed ibn Mûsâ al-Khowârizmî produced a work containing a treatment of algebra. To quote Cajori:

This is the first book known to contain this word [algebra] itself as title. Really the title consists of two words, *al-jebr w'almuqabala*, the nearest English translation of which is "restoration and reduction." By "restoration" was meant the transposing of negative terms to the other side of the equation; by "reduction," the uniting of similar terms. Thus, $x^2 - 2x = 5x + 6$ passes by *al-jebr* into $x^2 = 5x + 2x + 6$; and this, by *almuqabala*, into $x^2 = 7x + 6$.

Not only is there very little in this work that is original, but there is actually a retrogression from the best that the Hindus had done. The forms $ax^2 = bx$, $ax^2 = c$, $bx = c$, $x^2 + bx = c$, $x^2 + c = bx$, $x^2 = bx + c$ are given as separate cases, and there are rules for solving each case. That is, there is a return to the stage of Diophantus. The comparatively highly developed symbolism of the Hindus is abandoned for one similar to that of the Greeks.

Omar Khayyám.—(1044–1123) and others before him solved cubics by finding the intersection of conics, and Abû'l-Wefâ made attempts to solve certain biquadratics by this method. This was the greatest achievement of the Arabs in algebra, though it was not wholly original; the beginnings had been made by the Greeks, who constructed in this way the solution of the old problem of doubling the cube; that is, of finding one root of $x^3 = 2a^3$. It was generally believed by the Arabs that the cubic and biquadratic were impossible of solution by algebraic methods.

The Chinese.—We shall mention very briefly the chief attainments of the Chinese in algebra. In a manuscript dating from the third century A.D., which was possibly copied from one much earlier, problems are given leading to quadratic equations such as $x^2 + 34x - 35500 = 0$. From the vague rules given for finding the answer it cannot be said with certainty whether or not they were in possession of a method for solving quadratics. A similar remark applies to cubic equations which appeared in the seventh century. But in 1247, what they called the "celestial method" made a decided advance in the solution of numerical equations. It was nothing less than Horner's method applied to the equation $x^4 - 763200x^2 - 40642560000 = 0$ (note that this equation might be solved as a quadratic for x^2). But this greatest achievement in algebra of the Chinese was completely forgotten even in China, and knowledge of it never reached the outside world until long after the method had been invented and placed on a firm basis in the west.

The binomial coefficients for integral exponents were known and arranged in a triangle similar to the Pascal triangle (see page 293).

Western Europe

Ancient Rome does not have a place in the history of science. She contributed nothing to mathematics; and the primitive system of numerals which we know as Roman was not improved into something better—it was, in time, replaced. We have already noted that in fractions 12 was used as a preferred denominator. Other denominators in common use were 24, 48, and 72. On the Roman abacus there were columns for 12ths, 24ths, 48ths, and 72ds. There were in use a few simple rules for measuring areas (field measuring), but nothing was known of Greek geometry or algebra.

From Boethius to Leonardo of Pisa.—In 476 the Roman Empire finally fell and came under the sway of the Goths. With this event began a temporary interest in Greek science. *Boethius,* who died in 524, translated a Greek arithmetic (by Nicomachus) and extracts from Euclid's elements. This is of interest, since these translations constituted very nearly the total mathematical resources of the west down to the twelfth century. During these years the main mathematical interest of Christian Europe centered around the computing of Easter time. There was also instruction in finger reckoning and use of the abacus for business purposes. Names like ***Bede the Venerable*** (673–735) in England and ***Alcuin*** (735–804) who came to the court of Charlemagne may be mentioned. ***Gerbert,*** who died 1003 (as Pope Sylvester II), revived the study of mathematics but did not go beyond Boethius.

Translation from the Arabic began early in the twelfth century. The "Arabic numerals" were brought into use, but with surprising and persistent opposition. ***Athelhard of Bath,*** about 1125, traveled in Spain, Egypt, and the eastern part of the Mohammedan empire, and brought Arab works to the attention of the West. ***John of Seville*** (about 1150) showed Arab influence and did not use the abacus.

Leonardo of Pisa (***Fibonacci,*** 1170–1250) is the first important name in western European mathematics. He advocated the adoption of the "Arabic numerals." He studied the quadratic in Arabic sources and like the Arabs recognized but rejected negative solutions. He studied the cubic $x^3 + 2x^2 + 10x = 20$ and found a close approximation to one of its roots, but we do not know what method he used. After Leonardo ensued three centuries of stagnation. It was a period of wars, religious intolerance, mysticism, and what we might be inclined to call silly philosophical wrangling. "How many angels could occupy the point of a needle?" The number 6 is perfect $(1 + 2 + 3 = 6)$ and the creation of the world took place in six days. The number 666 was the number of the "beast." Such material gave rise to great treatises.

Solution of the cubic and biquadratic.—With the Renaissance, the rebirth of learning, western Europe literally burst into intellectual life and with it came also a new interest in mathematics. Brilliant successes were made at the very start. The first scene is laid in Italy. ***Scipione del Ferro*** (1465–1526) asserted he had solved the cubic $x^3 + mx = n$, but did not publish his work. In 1530 ***Nicolo of***

Brescia, usually called *Tartaglia* (the stammerer), said he had found a method for solving $x^3 + px^2 = q$. *Floridus*, a pupil of del Ferro, proclaimed that he had solved $x^3 + mx = n$, presumably having learned the solution from his master. In those days, and for long after, it was common practice to keep such discoveries secret and to challenge rivals to establish title to championships. Tartaglia challenged Floridus, the contest being set for February 22, 1535. To prepare for this contest Tartaglia went to work in earnest and succeeded in finding solutions for the various forms in which they put the cubic. Tartaglia won "hands down," solving the whole set of problems in two hours, while Floridus did not solve any. Tartaglia was working on translations of Euclid and Archimedes and intended to write a great work of his own in which his solution of the cubic should appear as the crowning element. At this time *Girolamo Cardano* (1501–1576), commonly called Cardan, was working on his great book (*Ars Magna*). He endeavored by all the wiles at his command to learn Tartaglia's secret and finally under the most solemn pledge from Cardan not to reveal it Tartaglia imparted it to him. Cardan blandly disregarded his pledge and published the solution in his *Ars Magna* (1545), mentioning, however, that he had it from his "friend Tartaglia." The solution usually called Cardan's solution certainly is not due to him. The only question is as to whether it should be assigned to del Ferro or to Tartaglia. From the result of the contest between Tartaglia and Floridus it seems fair to assume that the latter had not mastered the solution and inasmuch as the only communication made of the alleged discovery by del Ferro was the imparting of it to Floridus, it seems equally fair that Tartaglia should be credited with the solution.

Lodovico Ferrari (1522–1560), a pupil of Cardan, solved the equation $x^4 + 6x^2 + 30 = 60x$ by the method given in Chapter 19 of this book. He also solved other biquadratics by the same method. The solution of the third and fourth degree equation was a sudden and tremendous advance. The complete solution of the quadratic had been perfected at least one thousand years earlier, and, as noted above, the Arabs had come to the conclusion that equations of higher degree could not be solved by algebraic methods. Great interest in the solution of equations of degrees higher than the fourth was now created and continued for nearly three hundred years. But all efforts failed and in 1824 ***Abel*** proved that a general equation of the fifth degree or higher cannot be solved by ordinary algebraic processes.

Theory of equations.—The theorems on the theory of equations given in Chapters 17 and 18 of this book came to light at various times, all of them in western Europe. ***Vieta*** (1540-1603) noted the relation between the coefficients and the roots of a quadratic in case the roots are both positive. Later it was noted (***Peletier,*** 1517–1582) that a root is always a divisor of the last term. ***Girard*** (1595–1632) inferred by induction that the number of roots in an equation is equal to its degree, and he also obtained more general relations between roots and coefficients. ***René Descartes*** stated in incomplete form the "rule of signs" ascribed to him (see page 246), and ***Sir Isaac Newton*** (1642–1727) restated it in accurate form. Proofs of this rule were made in the eighteenth century and ***Gauss*** stated and proved the more precise rule given on page 247. Newton also gave a method for establishing the upper limit of the roots. ***Joseph Louis Lagrange*** (1736–1813) gave a proof that every equation has a root. This theorem was proved later by others including Gauss. The methods for changing signs of roots, diminishing or increasing them, and multiplying them are in reality very simple and were found incidentally by competent workers while developing a method for approximating roots of numerical equations.

Approximating roots of numerical equations.—As was noted above, the Chinese in the thirteenth century approximated roots of numerical equations, using practically what we call Horner's method (Chapter 17). Not only did this method fail to influence the development of algebra outside of China, but the Chinese themselves forgot it completely. Leonardo of Pisa approximated a root of a cubic by an unknown method; Cardan worked on the same problem, using the method of false position; Vieta made approximations by a more complex method which we shall not describe; and Newton solved the problem, using essentially the method of Example 1,

p. 242. This last method may fail in some exceptional cases and of this Newton was aware. A modification of Newton's method was made by one of his friends, ***Ralphson,*** who used the derivative. This method is often referred to as Newton's. ***Horner's method,*** discovered by ***W. G. Horner,*** an Englishman, was published in 1819. A similar method had been given earlier (1802) by the Italian ***Paolo Ruffini.*** Both Horner and Ruffini offered their methods as substitutes for the old process of extracting roots of numbers. Ruffini's publication was neglected and forgotten and Horner did not know of it; and neither of them, of course, knew of the Chinese "celestial method." Horner's method as now given in texts is a somewhat simplified form of his original method.

Determinants.—As would naturally be the case, determinants were discovered in connection with the solution of simultaneous linear equations. The original invention is usually credited to ***Gottfried Wilhelm Leibniz*** (1646–1716), who (1693) gave a relation which must exist among the coefficients of three linear equations in two unknowns in order that the equations shall be consistent. This relation among the coefficients (see §236) is that what we now call a determinant shall be equal to zero. Leibniz used determinants also to simplify elimination in working with linear equations. Though it had no effect upon the development of this subject in the west, it is of interest to note that the Japanese mathematician ***Seki*** (1642–1708) considered determinants somewhat earlier than Leibniz. Seki dealt with n equations instead of three. "He knew that a determinant of the nth order has $n!$ terms when expanded and that rows and columns can be interchanged." The theory of determinants is very extensive and has many ramifications. Some of the greatest names in the history of modern mathematics are associated with its development; ***Laplace, Cauchy, Jacobi,*** and ***Cayley*** may be mentioned. To Cayley (1821–1895) is due the present method of writing the determinant by placing the square array between two vertical lines.

Development of the algebraic number system.—The idea of signed or directed numbers was surprisingly slow in securing general recognition. Newton was one of the first to use letters to represent constants and to let these letters represent any numbers, positive or negative. Tartaglia considered the "two types" of equations $x^3 + mx = n$ and $x^3 = mx + n$ which means that m was a positive number only. Cardan had arrived at about the same point in this respect as the ancient Hindus. Descartes in his analytic geometry admitted negative values of a function but not of the independent variable; that is, he worked in the first and fourth quadrants only. Newton worked freely in all four quadrants.

Complex numbers were noticed but not taken seriously as numbers; the story is much the same as that of signed numbers, though it falls later. ***Léonard Euler*** (1707–1783) and others elaborated the theory of complex numbers, but they were not formally accepted until a geometric representation was found. To quote Cajori:

> The earliest printed graphic representation of $\sqrt{-1}$ and $a + b\sqrt{-1}$ was given in an "Essay on the Analytic Representation of Direction, with Applications in Particular to the Determination of Plane and Spherical Polygons," presented in 1797 by Caspar Wessel (1745–1818) to the Royal Academy of Sciences and Letters of Denmark and published in Vol. V of its Memoirs in 1799. . . . In 1897 a French translation was brought out by the Danish Academy. Another noteworthy publication which remained unknown for many years is an Essay published in 1806 by Jean Robert Argand (1768–1822) of Geneva, containing a geometric representation of $a + \sqrt{-1}b$. Some parts of his paper are less rigorous than the corresponding parts of Wessel. Argand gave some remarkable applications to trigonometry, geometry, and algebra. The word "modulus," to represent the length of the vector $a + ib$, is due to Argand. The writings of Wessel and Argand being little noticed, it remained for K. F. Gauss to break down the last opposition to the imaginary. Gauss seems to have been in possession of a graphic scheme as early as 1799, but its fuller exposition was deferred until 1831.

Development of modern notation and terminology.—With the standard histories of mathematics it is easy enough to find at what time and by whom the various symbols of algebra are believed to have been first used. But this would not tell the whole story. Symbols came and went and came again. For long periods there were rival symbols for the same purpose. The present almost uniform international symbolism for algebra is the result of a long struggle for survival. The following is from Cajori:

Leonardo of Pisa possessed no algebraic symbolism. Like the early Arabs, he expressed the relations of magnitudes to each other by lines or in words. . . . The most epoch-making innovation in algebra due to Vieta is the denoting of general or indefinite quantities by letters of the alphabet. . . . The equation $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$ was written by him "a cubus + b in a quadr. 3 + a in b quadr. 3 + b cubo aequalia $\overline{a + b}$ cubo." In numerical equations the unknown quantity was denoted by N, its square by Q, and its cube by C. Thus the equation $x^3 - 8x^2 + 16x = 40$ was written $1C - 8Q + 16N$ aequal 40. Vieta used the term "coefficient," but it was little used before the close of the seventeenth century. Sometimes he uses also the term "polynomial." Observe that exponents and our symbol (=) for equality were not yet in use; but that Vieta employed the Maltese cross (+) as the short hand symbol for addition, and the (−) for subtraction. These two characters had not been in very general use before the time of Vieta. "It is very singular," says Hallam, "that discoveries of the greatest convenience, and, apparently, not above the ingenuity of a village school master, should have been overlooked by men of extraordinary acuteness like Tartaglia, Cardan, and L. Ferrari; and, hardly less so that, by dint of that acuteness, they dispensed with the aid of these contrivances in which we suppose that so much of the utility of algebraic expression consists." Even after improvements in notation were once proposed, it was with extreme slowness that they were admitted into general use. They were made oftener by accident than design, and their authors had little notion of the effect of the change which they were making. The introduction of the + and − symbols seems to be due to the Germans, who, although they did not enrich algebra during the Renaissance with great inventions, as did the Italians, still cultivated it with great zeal. The arithmetic of John Widmann, brought out in 1489 in Leipzig, is the earliest printed book in which the + and − symbols have been found. The + sign is not restricted by him to ordinary addition; it has the more general meaning "et" or "and" as in the heading, "regula augmenti + decrementi." The − sign is used to indicate subtraction, but not regularly so. The word "plus" does not occur in Widmann's text; the word "minus" is used only two or three times. . . . Christoff Rudolff, in his algebra, remarks that "the *radix quadrata* is, for brevity, designated in his logarithm with the character $\surd$, as $\surd 4$." Here the dot has grown into a symbol much like our own. This same symbol was used by Michael Stifel. Our sign of equality is due to Robert Recorde (1510–1558), the author of The Whetstone of Witte (1557), which is the first English treatise on algebra. He selected this symbol because no two things could be more equal than two parallel lines =. The sign ÷ for division was first used by Johann Heinrich Rahn, a Swiss, in his Teutsche Algebra, Zurich, 1659, and was introduced in England through Thomas Brancker's translation of Rahn's book, London, 1668. . . .

Harriot made some changes in algebraic notation, adopting small letters of the alphabet in place of the capitals used by Vieta. The symbols of inequality > and < were introduced by him. The signs ≦ and ≧ were first used about a century later by the Parisian hydrographer, Pierre Bouguer. . . . Oughtred laid extraordinary emphasis upon the use of mathematical symbols; altogether he used over 150 of them. Only three have come down to modern times, namely, × as the symbol of multiplication, :: as that of proportion, and ∼ as that for "difference." The symbol × occurs in the Clavis, but the letter X which closely resembles it, occurs as a sign of multiplication in the anonymous "Appendix to the Logarithmes" in Edward Wright's translation of Napier's Descriptio, published in 1618. . . . It is interesting to note the attitude of Leibniz toward some of these symbols. On July 29, 1698, he wrote in a letter to John Bernoulli: "I do not like × as a symbol for multiplication, as it is easily confounded with x; . . . often I simply relate two quantities by an interposed dot and indicate multiplication by $ZC \cdot LM$. . . ." Oughtred did not use parentheses. Terms to be aggregated were enclosed between double colons. He wrote $\sqrt{(A + E)}$ thus, $\sqrt{q}:A + E:$ The two dots at the end were sometimes omitted. Thus, $C:A + B - E$ meant $(A + B - E)^3$. Before Oughtred the use of parentheses had been suggested by Clavius in 1608 and Girard in 1629. In fact, as early as 1556 Tartaglia wrote $\sqrt{\sqrt{28} - \sqrt{10}}$ thus, ℞ v. (℞28 men ℞10), where ℞ v. means "radix universalis," but he did not use parentheses in indicating the product of two expressions. Parentheses were used by I. Errard de Bar le Duc (1619), Jacobo de Billy (1643), Richard Norwood (1631), Samuel Foster (1659); nevertheless, parentheses did not become popular in algebra before the time of Leibniz and the Bernoullis.

Miscellaneous topics.—To trace even briefly the story of the many topics not yet considered would require much space. The extension of the "Arabic notation" to include decimal fractions was a very slow process. Their full use, both as a notation for numbers and in computation, came first with Napier, the inventor of logarithms. The idea of infinite series and their convergence came slowly.

Pietro Mengoli (1626–1686) proved the harmonic series divergent (see page 315). Newton developed the binomial formula for fractional exponents and thus obtained an infinite series. Newton also introduced the system of literal indices. ***Edward Waring*** (1734–1798) knew that $1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$ is convergent when $n > 1$ and used the ratio test for the convergence of series.

Logarithms were invented by *John Napier*, Baron of Merchiston, Scotland (1550–1617), and a table was published in 1614. Napier's logarithms were made to depend upon a correspondence between an arithmetic and a geometric series, and were in no way related to exponents. The latter came into use many years later. It was only in the time of Euler, about 1750, that logarithms came to be regarded as exponents.

Napier's tables differed in many respects from those we now use. His friend *Briggs* constructed a table using the base 10, the general plan for the Briggsian table being the joint product of Napier and Briggs. Napier died in 1617, but Briggs in 1624 published his Arithmetica Logarithmica containing 14 place logarithms of numbers from 1 to 20,000 and from 90,000 to 100,000. The gap was filled by *Adrian Vlacq*, who was born in Holland and lived in London and Paris. In 1628 Vlacq published the complete table up to 100,000, 70,000 of which were computed by himself. Says Cajori:

Briggs and Vlacq published four fundamental works, the results of which have not been superseded by any subsequent calculations until very recently.

The word "characteristic," as used in logarithms, first occurs in Briggs' Arithmetica Logarithmica, 1624; the word "mantissa" was introduced by John Wallis, 1693.

Briggs divided a degree into 100 parts, as was done also by N. Roe in 1633, W. Oughtred in 1657, and John Newton in 1658, but owing to the publication by Vlacq of trigonometrical tables constructed on the old sexagesimal division, Briggs' innovation did not prevail.

Edmund Gunter of London found the logarithmic sines and tangents for every minute to seven places. He was the inventor of the words cosine and cotangent.

Abraham de Moivre (1667–1754), the author of the theorem known by his name, was of French descent and lived in London, where he enjoyed the high respect of the mathematicians of his time.

Newton himself, in the later years of his life, used to reply to inquiries respecting mathematics, even respecting his Principia: "Go to Mr. De Moivre: he knows these things better than I do." . . . Shortly before his death he declared that it was necessary for him to sleep ten or twenty minutes longer every day. The day after he had reached the total of over twenty-three hours, he slept exactly twenty-four hours and then passed away in his sleep. [Cajori.]

A college freshman remarked that de Moivre died of an arithmetic progression.

The theory of probability had its origin in studying chances in gambling. Cardan was an inveterate gambler and contributed to the theory. "A correspondence between Blaise Pascal and Fermat relating to a certain game of chance was the germ of the theory of probability, of which certain anticipations are found in Cardan, Tartaglia, Kepler, and Galileo."

Connected with the theory of probability were the investigations on mortality and insurance. The use of tables of mortality does not seem to have been altogether unknown to the ancients, but the first name usually mentioned in this connection is Captain John Graunt who published at London in 1662 his Natural and Political Observations . . . made upon the *bills of mortality*, basing his deductions upon records of deaths which began to be kept in London in 1592 and were first intended to make known the progress of the plague. [Cajori.]

This theory has been studied extensively in the last two centuries.

Amount of $1 at Compound Interest. $(1+i)^n$.

n	1%	1½%	2%	3%	4%	5%	6%	7%
1	1.0100	1.0150	1.0200	1.0300	1.0400	1.0500	1.0600	1.0700
2	1.0201	1.0302	1.0404	1.0609	1.0816	1.1025	1.1236	1.1449
3	1.0303	1.0457	1.0612	1.0927	1.1249	1.1576	1.1910	1.2250
4	1.0406	1.0614	1.0824	1.1255	1.1699	1.2155	1.2625	1.3108
5	1.0510	1.0773	1.1041	1.1593	1.2167	1.2763	1.3382	1.4026
6	1.0615	1.0934	1.1262	1.1941	1.2653	1.3401	1.4185	1.5007
7	1.0721	1.1098	1.1487	1.2299	1.3159	1.4071	1.5036	1.6058
8	1.0829	1.1265	1.1717	1.2668	1.3686	1.4775	1.5938	1.7182
9	1.0937	1.1434	1.1951	1.3048	1.4233	1.5513	1.6895	1.8385
10	1.1046	1.1605	1.2190	1.3439	1.4802	1.6289	1.7908	1.9672
11	1.1157	1.1779	1.2434	1.3842	1.5395	1.7103	1.8983	2.1049
12	1.1268	1.1956	1.2682	1.4258	1.6010	1.7959	2.0122	2.2522
13	1.1381	1.2136	1.2936	1.4685	1.6651	1.8856	2.1329	2.4098
14	1.1495	1.2318	1.3195	1.5126	1.7317	1.9799	2.2609	2.5785
15	1.1610	1.2502	1.3459	1.5580	1.8009	2.0789	2.3966	2.7590
16	1.1726	1.2690	1.3728	1.6047	1.8730	2.1829	2.5404	2.9522
17	1.1843	1.2880	1.4002	1.6528	1.9479	2.2920	2.6928	3.1588
18	1.1961	1.3073	1.4282	1.7024	2.0258	2.4066	2.8543	3.3799
19	1.2081	1.3270	1.4568	1.7535	2.1068	2.5270	3.0256	3.6165
20	1.2202	1.3469	1.4859	1.8061	2.1911	2.6533	3.2071	3.8697
21	1.2324	1.3671	1.5157	1.8603	2.2788	2.7860	3.3996	4.1406
22	1.2447	1.3876	1.5460	1.9161	2.3699	2.9253	3.6035	4.4304
23	1.2572	1.4084	1.5769	1.9736	2.4647	3.0715	3.8197	4.7405
24	1.2697	1.4295	1.6084	2.0328	2.5633	3.2251	4.0489	5.0724
25	1.2824	1.4509	1.6406	2.0938	2.6658	3.3864	4.2919	5.4274
26	1.2953	1.4727	1.6734	2.1566	2.7725	3.5557	4.5494	5.8074
27	1.3082	1.4948	1.7069	2.2213	2.8834	3.7335	4.8223	6.2139
28	1.3213	1.5172	1.7410	2.2879	2.9987	3.9201	5.1117	6.6488
29	1.3345	1.5400	1.7758	2.3566	3.1187	4.1161	5.4184	7.1143
30	1.3478	1.5631	1.8114	2.4273	3.2434	4.3219	5.7435	7.6123
31	1.3613	1.5865	1.8476	2.5001	3.3731	4.5380	6.0881	8.1451
32	1.3749	1.6103	1.8845	2.5751	3.5081	4.7649	6.4534	8.7153
33	1.3887	1.6345	1.9222	2.6523	3.6484	5.0032	6.8406	9.3253
34	1.4026	1.6590	1.9607	2.7319	3.7943	5.2533	7.2510	9.9781
35	1.4166	1.6839	1.9999	2.8139	3.9461	5.5160	7.6861	10.6766
36	1.4308	1.7091	2.0399	2.8983	4.1039	5.7918	8.1473	11.4239
37	1.4451	1.7348	2.0807	2.9852	4.2681	6.0814	8.6361	12.2236
38	1.4595	1.7608	2.1223	3.0748	4.4388	6.3855	9.1543	13.0793
39	1.4741	1.7872	2.1647	3.1670	4.6164	6.7048	9.7035	13.9948
40	1.4889	1.8140	2.2080	3.2620	4.8010	7.0400	10.2857	14.9745
41	1.5038	1.8412	2.2522	3.3599	4.9931	7.3920	10.9029	16.0227
42	1.5188	1.8688	2.2972	3.4607	5.1928	7.7616	11.5570	17.1443
43	1.5340	1.8969	2.3432	3.5645	5.4005	8.1497	12.2505	18.3444
44	1.5493	1.9253	2.3901	3.6715	5.6165	8.5572	12.9855	19.6285
45	1.5648	1.9542	2.4379	3.7816	5.8412	8.9850	13.7646	21.0025
46	1.5805	1.9835	2.4866	3.8950	6.0748	9.4343	14.5905	22.4726
47	1.5963	2.0133	2.5363	4.0119	6.3178	9.9060	15.4659	24.0457
48	1.6122	2.0435	2.5871	4.1323	6.5705	10.4013	16.3939	25.7289
49	1.6283	2.0741	2.6388	4.2562	6.8333	10.9213	17.3775	27.5299
50	1.6446	2.1052	2.6916	4.3839	7.1067	11.4674	18.4202	29.4570

Amount of an Annuity of \$1. $s_n = \dfrac{(1+i)^n - 1}{i}$

n	1%	1½%	2%	3%	4%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0150	2.0200	2.0300	2.0400	2.0500	2.0600	2.0700
3	3.0301	3.0452	3.0604	3.0909	3.1216	3.1525	3.1836	3.2149
4	4.0604	4.0909	4.1216	4.1836	4.2465	4.3101	4.3746	4.4399
5	5.1010	5.1523	5.2040	5.3091	5.4163	5.5256	5.6371	5.7507
6	6.1520	6.2296	6.3081	6.4684	6.6330	6.8019	6.9753	7.1533
7	7.2135	7.3230	7.4343	7.6625	7.8983	8.1420	8.3938	8.6540
8	8.2857	8.4328	8.5830	8.8923	9.2142	9.5491	9.8975	10.2598
9	9.3685	9.5593	9.7546	10.1591	10.5828	11.0266	11.4913	11.9780
10	10.4622	10.7027	10.9497	11.4639	12.0061	12.5779	13.1808	13.8164
11	11.5668	11.8633	12.1687	12.8078	13.4864	14.2068	14.9716	15.7836
12	12.6825	13.0412	13.4121	14.1920	15.0258	15.9171	16.8699	17.8885
13	13.8093	14.2368	14.6803	15.6178	16.6268	17.7130	18.8821	20.1406
14	14.9474	15.4504	15.9739	17.0863	18.2919	19.5986	21.0151	22.5505
15	16.0969	16.6821	17.2934	18.5989	20.0236	21.5786	23.2760	25.1290
16	17.2579	17.9324	18.6393	20.1569	21.8245	23.6575	25.6725	27.8881
17	18.4304	19.2014	20.0121	21.7616	23.6975	25.8404	28.2129	30.8402
18	19.6147	20.4894	21.4123	23.4144	25.6454	28.1324	30.9057	33.9990
19	20.8109	21.7967	22.8406	25.1169	27.6712	30.5390	33.7600	37.3790
20	22.0190	23.1237	24.2974	26.8704	29.7781	33.0660	36.7856	40.9955
21	23.2392	24.4705	25.7833	28.6765	31.9692	35.7193	39.9927	44.8652
22	24.4716	25.8376	27.2990	30.5368	34.2480	38.5052	43.3923	49.0057
23	25.7163	27.2251	28.8450	32.4529	36.6179	41.4305	46.9958	53.4361
24	26.9735	28.6335	30.4219	34.4265	39.0826	44.5020	50.8156	58.1767
25	28.2432	30.0630	32.0303	36.4593	41.6459	47.7271	54.8645	63.2490
26	29.5256	31.5140	33.6709	38.5530	44.3117	51.1135	59.1564	68.6765
27	30.8209	32.9867	35.3443	40.7096	47.0842	54.6691	63.7058	74.4838
28	32.1291	34.4815	37.0512	42.9309	49.9676	58.4026	68.5281	80.6977
29	33.4504	35.9987	38.7922	45.2189	52.9663	62.3227	73.6398	87.3465
30	34.7849	37.5387	40.5681	47.5754	56.0849	66.4388	79.0582	94.4608
31	36.1327	39.1018	42.3794	50.0027	59.3283	70.7608	84.8017	102.0730
32	37.4941	40.6883	44.2270	52.5028	62.7015	75.2988	90.8898	110.2182
33	38.8690	42.2986	46.1116	55.0778	66.2095	80.0638	97.3432	118.9334
34	40.2577	43.9331	48.0338	57.7302	69.8579	85.0670	104.1838	128.2588
35	41.6603	45.5921	49.9945	60.4621	73.6522	90.3203	111.4348	138.2369
36	43.0769	47.2760	51.9944	63.2759	77.5983	95.8363	119.1209	148.9135
37	44.5076	48.9851	54.0343	66.1742	81.7022	101.6281	127.2681	160.3374
38	45.9527	50.7199	56.1149	69.1594	85.9703	107.7095	135.9042	172.5610
39	47.4123	52.4807	58.2372	72.2342	90.4091	114.0950	145.0585	185.6403
40	48.8864	54.2679	60.4020	75.4013	95.0255	120.7998	154.7620	199.6351
41	50.3752	56.0819	62.6100	78.6633	99.8265	127.8398	165.0477	214.6096
42	51.8790	57.9231	64.8622	82.0232	104.8196	135.2318	175.9505	230.6322
43	53.3978	59.7920	67.1595	85.4839	110.0124	142.9933	187.5076	247.7765
44	54.9318	61.6889	69.5027	89.0484	115.4129	151.1430	199.7580	266.1209
45	56.4811	63.6142	71.8927	92.7199	121.0294	159.7002	212.7435	285.7493
46	58.0459	65.5684	74.3306	96.5015	126.8706	168.6852	226.5081	306.7518
47	59.6263	67.5519	76.8172	100.3965	132.9454	178.1194	241.0986	329.2244
48	61.2226	69.5652	79.3535	104.4084	139.2632	188.0254	256.5645	353.2701
49	62.8348	71.6087	81.9406	108.5406	145.8337	198.4267	272.9584	378.9990
50	64.4632	73.6828	84.5794	112.7969	152.6671	209.3480	290.3359	406.5289

Present Value of an Annuity of \$1. $a_{\overline{n}|} = \dfrac{1-(1+i)^{-n}}{i}$

n	1%	1½%	2%	3%	4%	5%	6%	[illegible]
1	0.9901	0.9852	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346
2	1.9704	1.9559	1.9416	1.9135	1.8861	1.8594	1.8334	1.8080
3	2.9410	2.9122	2.8839	2.8286	2.7751	2.7232	2.6730	2.6243
4	3.9020	3.8544	3.8077	3.7171	3.6299	3.5460	3.4651	3.3872
5	4.8534	4.7826	4.7135	4.5797	4.4518	4.3295	4.2124	4.1002
6	5.7955	5.6972	5.6014	5.4172	5.2421	5.0757	4.9173	4.7665
7	6.7282	6.5982	6.4720	6.2303	6.0021	5.7864	5.5824	5.3893
8	7.6517	7.4859	7.3255	7.0197	6.7327	6.4632	6.2098	5.9713
9	8.5660	8.3605	8.1622	7.7861	7.4353	7.1078	6.8017	6.5152
10	9.4713	9.2222	8.9826	8.5302	8.1109	7.7217	7.3601	7.0236
11	10.3676	10.0711	9.7868	9.2526	8.7605	8.3064	7.8869	7.4987
12	11.2551	10.9075	10.5753	9.9540	9.3851	8.8633	8.3838	7.9427
13	12.1337	11.7315	11.3484	10.6350	9.9856	9.3936	8.8527	8.3577
14	13.0037	12.5434	12.1062	11.2961	10.5631	9.8986	9.2950	8.7455
15	13.8651	13.3432	12.8493	11.9379	11.1184	10.3797	9.7122	9.1079
16	14.7179	14.1313	13.5777	12.5611	11.6523	10.8378	10.1059	9.4466
17	15.5623	14.9076	14.2919	13.1661	12.1657	11.2741	10.4773	9.7632
18	16.3983	15.6726	14.9920	13.7535	12.6593	11 6896	10.8276	10.0591
19	17.2260	16.4262	15.6785	14.3238	13.1339	12.0853	11.1581	10.3356
20	18.0456	17.1686	16.3514	14.8775	13.5903	12.4622	11.4699	10.5940
21	18.8570	17.9001	17.0112	15.4150	14.0292	12.8212	11.7641	10.8355
22	19.6604	18.6208	17.6580	15.9369	14.4511	13.1630	12.0416	11.0612
23	20.4558	19.3309	18.2922	16.4436	14.8568	13.4886	12.3034	11.2722
24	21.2434	20.0304	18.9139	16.9355	15.2470	13.7986	12.5504	11.4693
25	22.0232	20.7196	19.5235	17.4131	15.6221	14.0939	12.7834	11.6536
26	22.7952	21.3986	20.1210	17.8768	15.9828	14.3752	13.0032	11.8258
27	23.5596	22.0676	20.7069	18.3270	16.3296	14.6430	13.2105	11.9867
28	24.3164	22.7267	21.2813	18.7641	16.6631	14.8981	13.4062	12.1371
29	25.0658	23.3761	21.8444	19.1885	16.9837	15.1411	13.5907	12.2777
30	25.8077	24.0158	22.3965	19.6004	17.2920	15.3725	13.7648	12.4090
31	26.5423	24.6461	22.9377	20.0004	17.5885	15.5928	13.9291	12.5318
32	27.2696	25.2671	23.4683	20.3888	17.8736	15.8027	14.0840	12.6466
33	27.9897	25.8790	23.9886	20.7658	18.1476	16.0025	14.2302	12.7538
34	28.7027	26.4817	24.4986	21.1318	18.4112	16.1929	14.3681	12.8540
35	29.4086	27.0756	24.9986	21.4872	18.6646	16.3742	14.4982	12.9477
36	30.1075	27.6607	25.4888	21.8323	18.9083	16.5469	14.6210	13.0352
37	30.7995	28.2371	25.9695	22.1672	19.1426	16.7113	14.7368	13.1170
38	31.4847	28.8051	26.4406	22.4925	19.3679	16.8679	14.8460	13.1935
39	32.1630	29.3646	26.9026	22.8082	19.5845	17.0170	14.9491	13.2649
40	32.8347	29.9158	27.3555	23.1148	19.7928	17.1591	15.0463	13.3317
41	33.4997	30.4590	27.7995	23.4124	19.9931	17.2944	15.1380	13.3941
42	34.1581	30.9941	28.2348	23.7014	20.1856	17.4232	15.2245	13.4524
43	34.8100	31.5212	28.6616	23.9819	20.3708	17.5459	15.3062	13.5070
44	35.4555	32.0406	29.0800	24.2543	20.5488	17.6628	15.3832	13.5579
45	36.0945	32.5523	29.4902	24.5187	20.7200	17.7741	15.4558	13.6055
46	36.7272	33.0565	29.8923	24.7754	20.8847	17.8801	15.5244	13.6500
47	37.3537	33.5532	30.2866	25.0247	21.0429	17.9810	15.5890	13.6916
48	37.9740	34.0426	30.6731	25.2667	21.1951	18.0772	15.6500	13.7305
49	38.5881	34.5247	31.0521	25.5017	21.3415	18.1687	15.7076	13.7668
50	39.1961	34.9997	31.4236	25.7298	21.4822	18.2559	15.7619	13.8007

Logarithms

n	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Logarithms

n	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Powers and Roots

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	60	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	70	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	80	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	90	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	100	10 000	1 000 000	10.000	4.642

Present Value of $1 at Compound Interest. $(1+i)^{-n}$.

n	1%	1¼%	2%	3%	4%	5%	6%	7%
1	0.9901	0.9852	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346
2	0.9803	0.9707	0.9612	0.9426	0.9246	0.9070	0.8900	0.8734
3	0.9706	0.9563	0.9423	0.9151	0.8890	0.8638	0.8396	0.8163
4	0.9610	0.9422	0.9238	0.8885	0.8548	0.8227	0.7921	0.7629
5	0.9515	0.9283	0.9057	0.8626	0.8219	0.7835	0.7473	0.7130
6	0.9420	0.9145	0.8880	0.8375	0.7903	0.7462	0.7050	0.6663
7	0.9327	0.9010	0.8706	0.8131	0.7599	0.7107	0.6651	0.6227
8	0.9235	0.8877	0.8535	0.7894	0.7307	0.6768	0.6274	0.5820
9	0.9143	0.8746	0.8368	0.7664	0.7026	0.6446	0.5919	0.5439
10	0.9053	0.8617	0.8203	0.7441	0.6756	0.6139	0.5584	0.5083
11	0.8963	0.8489	0.8043	0.7224	0.6496	0.5847	0.5268	0.4751
12	0.8874	0.8364	0.7885	0.7014	0.6246	0.5568	0.4970	0.4440
13	0.8787	0.8240	0.7730	0.6810	0.6006	0.5303	0.4688	0.4150
14	0.8700	0.8118	0.7579	0.6611	0.5775	0.5051	0.4423	0.3878
15	0.8613	0.7999	0.7430	0.6419	0.5553	0.4810	0.4173	0.3624
16	0.8528	0.7880	0.7284	0.6232	0.5339	0.4581	0.3936	0.3387
17	0.8444	0.7764	0.7142	0.6050	0.5134	0.4363	0.3714	0.3166
18	0.8360	0.7649	0.7002	0.5874	0.4936	0.4155	0.3503	0.2959
19	0.8277	0.7536	0.6864	0.5703	0.4746	0.3957	0.3305	0.2765
20	0.8195	0.7425	0.6730	0.5537	0.4564	0.3769	0.3118	0.2584
21	0.8114	0.7315	0.6598	0.5375	0.4388	0.3589	0.2942	0.2415
22	0.8034	0.7207	0.6468	0.5219	0.4220	0.3418	0.2775	0.2257
23	0.7954	0.7100	0.6342	0.5067	0.4057	0.3256	0.2618	0.2109
24	0.7876	0.6995	0.6217	0.4919	0.3901	0.3101	0.2470	0.1971
25	0.7798	0.6892	0.6095	0.4776	0.3751	0.2953	0.2330	0.1842
26	0.7720	0.6790	0.5976	0.4637	0.3607	0.2812	0.2198	0.1722
27	0.7644	0.6690	0.5859	0.4502	0.3468	0.2678	0.2074	0.1609
28	0.7568	0.6591	0.5744	0.4371	0.3335	0.2551	0.1956	0.1504
29	0.7493	0.6494	0.5631	0.4243	0.3207	0.2429	0.1846	0.1406
30	0.7419	0.6398	0.5521	0.4120	0.3083	0.2314	0.1741	0.1314
31	0.7346	0.6303	0.5412	0.4000	0.2965	0.2204	0.1643	0.1228
32	0.7273	0.6210	0.5306	0.3883	0.2851	0.2099	0.1550	0.1147
33	0.7201	0.6118	0.5202	0.3770	0.2741	0.1999	0.1462	0.1072
34	0.7130	0.6028	0.5100	0.3660	0.2636	0.1904	0.1379	0.1002
35	0.7059	0.5939	0.5000	0.3554	0.2534	0.1813	0.1301	0.0937
36	0.6989	0.5851	0.4902	0.3450	0.2437	0.1727	0.1227	0.0875
37	0.6920	0.5764	0.4806	0.3350	0.2343	0.1644	0.1158	0.0818
38	0.6852	0.5679	0.4712	0.3252	0.2253	0.1566	0.1092	0.0765
39	0.6784	0.5595	0.4619	0.3158	0.2166	0.1491	0.1031	0.0715
40	0.6717	0.5513	0.4529	0.3066	0.2083	0.1420	0.0972	0.0668
41	0.6650	0.5431	0.4440	0.2976	0.2003	0.1353	0.0917	0.0624
42	0.6584	0.5351	0.4353	0.2890	0.1926	0.1288	0.0865	0.0583
43	0.6519	0.5272	0.4268	0.2805	0.1852	0.1227	0.0816	0.0545
44	0.6454	0.5194	0.4184	0.2724	0.1780	0.1169	0.0770	0.0509
45	0.6391	0.5117	0.4102	0.2644	0.1712	0.1113	0.0727	0.0476
46	0.6327	0.5042	0.4022	0.2567	0.1646	0.1060	0.0685	0.0445
47	0.6265	0.4967	0.3943	0.2493	0.1583	0.1009	0.0647	0.0416
48	0.6203	0.4894	0.3865	0.2420	0.1522	0.0961	0.0610	0.0389
49	0.6141	0.4821	0.3790	0.2350	0.1463	0.0916	0.0575	0.0363
50	0.6080	0.4750	0.3715	0.2281	0.1407	0.0872	0.0543	0.0339

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